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MOMENTS OF MODIFIED FACTORIAL SERIES DISTRIBUTION

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1. Introduction.

A method using finite difference operators has been recently introduced in order to express the moments of discrete probability distributions.

Berg (1974, 1976) has introduced a new class of discrete distributions, which he calls Factorial Series Distributions (FSD) and Gupta(1974) has introduced and studied a class of distributions called Modified Power Series Distributions (MPSD).

Janardan(1984) has published some of the ordinary and factorial moments of FSD and MPSD. For the sake of completeness, we shall define these classes here.

A discrete random variable(r. v) X is said to have a MPSD if its probability function (p. f.) is

$$P(x;\theta)=\frac{\{g(\theta)\}^x a(x)}{f(\theta)} \quad \text{for } x\in T,$$

where T is a countable set of integers and the series

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function

$$f(\theta)=\sum_{x\in T} a(x)\{g(\theta)\}^x,\ a(x)=\frac{1}{x!}\Delta^x f(\theta)>0 \text{ for } x\in T^c;$$

$f(\theta)$ and $g(\theta)$ are positive, finite, and differentiable.

A r.v. X is said to have a FSD if its p.f. is given by

$$P(x;\theta)=\frac{\theta^{(x)}a(x)}{f(\theta)},\ x\in T,$$

where $\theta^{(x)}=\theta(\theta-1)\cdots(\theta-x+1)$ and $f(\theta)=\sum_{x\in T}\theta^{(x)}a(x)$, $a(x)>0$ being free of θ for $x=0,1,\cdots$.

In this paper, we introduce another class of discrete distributions and provide a method of finding the moments using finite difference operators.

2. New class of discrete distributions

Following Gupta's approach, we introduce 2 new classes called Modified Factorial Series Distributions (MFSD) and discuss the moments. We define

$$(2.1)\quad P(x;\theta)=P_r(X=x)=\frac{\{g(\theta)\}^{(x)}a(x)}{f(\theta)},\ x\in T,$$

where the series function $f(\theta)=\sum_{x\in T}\{g(\theta)\}^{(x)}a(x)$, $a(x)=\Delta^x f(0)/x!$, $f(\theta)$ and $g(\theta)$ are nonzero and

$$\{g(\theta)\}^{(x)}=g(\theta)\{g(\theta)-1\}\cdots\{g(\theta)-x+1\}.$$

The p.f. (2.1) can be called a Descending Modified Factorial Series Distribution (Descending MFSD).

Similary, we define

$$(2.2) \qquad P(x;\theta)=P_r(X=x)=\frac{\{g(\theta)\}^{[x]}a(x)}{f(\theta)}, \quad x\in T,$$

where $f(\theta)=\sum_{x\in T}\{g(\theta)\}^{[x]}a(x)$, $f(\theta)$ and $g(\theta)$ are nonzero,

$$\{g(\theta)\}^{[x]}=g(\theta)\{g(\theta)+1\}\cdots\{g(\theta)+x-1\},$$

and

$$a(x)=\Delta^x f(0)/x!.$$

The (2.2) can be called a Ascending Modified Factorial Distribution(Ascending MFSD).

EXAMPLE 1.1. In the generalized hypergeometric distribution

$$P(x)=\frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$

$$P(x;\theta)=\theta^{(x)}b^{(n-x)}n^{(x)}/(\theta+b)^{(n)}x!, \text{ where } a=\theta.$$

EXAMPLE 1.2. The inverse polya distribution;

$$P(x)=\binom{-p/r}{k}\binom{-q/r}{x}k\Big/\binom{-1/r}{k+x}(k+x)$$

$$=(-q/r)^{(x)}(-p/r)^{(k)}\binom{k+x}{x}k\Big/(-1/r)^{(k+x)}(k+x).$$

$$P(x;\theta)=(-p\theta)^{(k)}(-q\theta)^{(x)}k\{(k+x)!\}/(-\theta)^{(x+x)}(k+x)k!\,x!,$$
where $1/r=\theta$.

These Examples 1-1 and 1-2 are Descending MFSD's.

EXAMPLE 1.3. The polya distribution;

$$P(x)=\binom{-p/r}{x}\binom{-q/r}{n-x}\Big/\binom{-1/r}{n}$$

$$=\binom{n}{x}(p/r)^{[x]}(q/r)^{[n-x]}/(1/r)^{[n]}.$$

$P(x;\theta)=(p\theta)^{[x_1]}(q\theta)^{[x_2]}(x_1+x_2)!\,/\,\theta^{[x_1+x_2]}x_1!x_2!$, where $1/r=\theta$, $x=x_1$ and $n-x=x_2$.

This Example 1.3 is Ascending MFSD.

3. Moments of MFSD

PROPOSITION 1.

(3.1) if $EX^r=\sum_{k=0}^{r} c_k\Delta^k 0^r$, $EX^{(k)}=k!c_k$.

(3.2) If $EX^r=\sum_{k=0}^{r} d_k\nabla^k 0^r$, $EX^{[k]}=k!d_k$.

The c_k's and d_k's are related to the descending and the ascending factorial moments.

(3.1) is given in Janardan (1984), (3.2) is obtained as

$$EX^r=\sum_{k=0}^{r} d_k\nabla^k 0^r$$

$$=\sum_{k=0}^{\infty}\frac{EX^{[k]}}{k!}\nabla^k 0^r$$

$$=\sum_{k=0}^{r} EX^{[k]}\nabla^k 0^r. \tag{3.3}$$

(3.2)-(3.3) is

$$\sum_{k=0}^{r}\left(d_k\frac{EX^{[k]}}{k!}\right)\nabla^k 0^r=0.$$

$$EX^{[k]}=k!\ d_k.$$

THEOREM 1. The moments of p.f. (2.1) are

$$(3.4)\qquad EX^r=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{(x)}\Delta^k f(\theta-k)}{f(\theta)}\ \frac{\Delta^k 0^r}{k!},$$

$$(3.5)\qquad EX^{(r)}=\frac{\{g(\theta)\}^{(r)}\Delta^r f(\theta-r)}{f(\theta)}.$$

PROOF. $EX^r=\sum_{x=0}^{\infty} P(x;\ \theta)$

$$=\sum_{x=0}^{\infty}\frac{\{g(\theta)\}^{(x)}a(x)}{f(\theta)}(1+\Delta^x)0^r$$

$$=\sum_{x=0}^{r}\sum_{x=k}^{\infty}\frac{\{g(\theta)\}^{(x)}}{f(\theta)}\ \frac{\Delta^x f(\theta)}{(x-k)\,!}\ \frac{\Delta^k 0^r}{k!}$$

$$=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{(k)}}{f(\theta)}\Delta^k\left\{\sum_{y=0}^{\infty}\frac{\{g(\theta-k)\}^{(y)}\Delta^y f(\theta)}{y!}\right\}\frac{\Delta^k 0^r}{k!}$$

$$=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{(k)}\ \Delta^k f(\theta-k)}{f(\theta)}\ \frac{\Delta^k 0^r}{k!}.$$

Using (3.3)

$$EX^{(r)}=\frac{\{g(\theta)\}^{(r)}\ \Delta^r f(\theta-r)}{f(\theta)}.$$

In the example 1-1, let $f(\theta)=(\theta+b)^{(n)}/b^{(n-x)}$, $g(\theta)=\theta$ and $a(x)=n^{(x)}/x!$.

THEOREM 2. If p.f. (2.2) are

$$(3.6)\qquad EX^r=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{[k]}\ \Delta^k f(\theta+k)}{f(\theta)}\cdot\frac{\Delta^k 0^r}{k!},$$

$$(3.7)\qquad EX^{[r]}=\frac{\{g(\theta)\}^{[r]}\ \Delta^r f(\theta+r)}{f(\theta)}.$$

PROOF. $EX^r=\sum_{x=0}^{\infty} x^r P(x:\ \theta)$

$$=\sum_{x=0}^{\infty}\frac{\{g(\theta)\}^{[x]}a(x)}{f(\theta)}(1+\Delta)^x 0^r$$

$$=\sum_{x=0}^{r}\sum_{x=k}^{\infty}\frac{\{g(\theta)\}^{[x]}}{f(\theta)}\frac{\Delta^{x}f(\theta)}{(x-k)\,!}\frac{\Delta^{k}0^{r}}{k!}$$

(let $x-k=y$)

$$=\sum_{k=1}^{r}\sum_{y=0}^{\infty}\frac{[g(\theta)]^{[y+k]}}{f(\theta)}\frac{\Delta^{y+k}f(\theta)}{y!}\frac{\Delta^{k}0^{r}}{k!}$$

$$=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{[k]}}{f(\theta)}\Delta^{k}\left\{\sum_{y=0}^{\infty}\frac{\{g(\theta)\}^{[y]}\Delta^{y}f(\theta)}{y!}\right\}\frac{\Delta^{k}0^{r}}{k!}$$

$$=\sum_{k=1}^{r}\frac{\{g(\theta)\}^{[k]}\Delta^{k}f(\theta+k)}{f(\theta)}\frac{\Delta^{k}0^{r}}{k!.}.$$

(3.6) can also be represented by using backward difference operator

$$(3.8)\qquad EX^{r}=\sum_{k=1}^{r}(-1)^{k+r}\frac{\{g(\theta)\}^{[k]}\Delta^{k}f(\theta+k)}{f(\theta)}\frac{\nabla^{k}0^{r}}{k!}.$$

Using (3.4), (3.5) and (3.6)

$$EX^{[r]}=\frac{\{g(\theta)\}^{[r]}\Delta^{r}f(\theta+r)}{f(\theta)}.$$

5. Moments of multivariate MFSD

A random vector $X=(x_1,x_2,\cdots,x_n)$ a m-variate Descending MFSD of its p.f. is given by

$$(4.1)\qquad P(X;\theta)=\frac{\{g(\theta_1)\}^{(x_1)}\cdots\{g(\theta_m)\}^{(x_m)}a(x_1,x_2,\cdots,x_m)}{f(\theta_1,\theta_2,\cdots,\theta_m)}$$

$$=\left\{\prod_{i=1}^{m}\frac{\{g(\theta_i)\}^{(x_i)}a(X)}{f(\theta)}\right\},$$

where $f(\theta)=f(\theta_1,\theta_2,\cdots,\theta_m)$, $a(X)=a(x_1,\cdots,x_m)=\Delta_1^{x_1}\cdots\Delta_m^{x_m}f(\theta)$

is independent of θ_i's and $i=1, 2, \cdots m$,

$$f(\theta)=\prod_{i=1}^{m}\{g(\theta_i)\}^{(x_i)}a(X), \quad \Delta_i \text{ operates only } \theta_i.$$

Similarly, we define a m-variate Ascending MFSD whose p. f. is given by

$$(4.2) \qquad P(X;\theta)=\frac{\{g(\theta_1)\}^{[x_1]}\cdots\{g(\theta_m)\}^{[x_m]}a(x_1,x_2,\cdots,x_m)}{f(\theta_1,\theta_2,\cdots\theta_m)}$$

$$=\left\{\prod_{i=1}^{m}\frac{\{g(\theta_i)\}^{[x_i]}a(X)}{f(\theta)}\right\}.$$

THEOREM 3. The moments of the m-variate Descending MFSD (4.1) are given by

$$(4.3) \qquad EX_1^{r_1}X_2^{r_2}\cdots X_m^{r_m}$$

$$=\sum_{k_i=1}^{r_i}\left[\left\{\prod_{i=1}^{m}\frac{\{g(\theta_i)^{(k_i)}\Delta_i^{k_i}f(\theta_i-k_i)\}}{f(\theta)}\right\}\left\{\prod_{i=1}^{m}\frac{\Delta_i^{k_i}0_i^{r_i}}{k_i!}\right\}\right].$$

$$(4.4) \qquad EX_1^{(r_1)}X_2^{(r_2)}\cdots X_m^{(r_m)}$$

$$=\prod_{i=1}^{m}\frac{\{g(\theta_i)\}^{(r_i)}\Delta_i^{r_i}f(\theta_i-r_i)}{f(\theta)}.$$

THEOREM 4. The moments of the m-variate Ascending MFSD (4.2) are given by

$$(4.5) \qquad EX_1^{r_1}X_2^{r_2}\cdots X_m^{r_m}$$

$$=\sum_{k_i=1}^{r_i}\left[\left\{\prod_{i=1}^{m}\frac{\{g(\theta_i)\}^{[k_i]}\Delta_i^{k_i}f(\theta_i+k_i)}{f(\theta)}\right\}\left\{\prod_{i=1}^{m}\frac{\Delta_i^{k_i}0_i^{r_i}}{k_i!}\right\}\right].$$

$$(4.6) \qquad EX_1^{[r_1]}X_2^{[r_2]}\cdots X_m^{[r_m]}$$

$$=\prod_{i=1}^{m}\frac{\{g(\theta_i)\}^{[r_i]}\Delta_i^{r_i}f(\theta_i+r_i)}{f(\theta)}.$$

In Example 1.2, let $f(\theta)=(-\theta)^{(k+x)}$ and in Example 1.3 $f(\theta)=(-\theta)^{[x_1+x_2]}$. Both are 2-variate case.

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