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ON THE ASYMPTOTIC BEHAVIOR OF RESOLVENTS OF ACCRETIVE OPERATORS IN BANACH SPACES

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1. Introduction

Let $(E, \|\ \|)$ be a real Banach space and let I denote the identity. Recall that an operator $A \subset E \times E$ with domain $D(A)$ and range $R(A)$ is said to be *accretive* if $\|x_1 - x_2\| \leqq \|x_1 - x_2 + r(y_1 - y_2)\|$ for all $y_i \in Ax_i$, $i = 1, 2$, and $r > 0$. An accretive operator $A \subset E \times E$ is *m-accretive* if $R(I + rA) = E$ for all $r > 0$. Let $J_t = (I + tA)^{-1}$, $t > 0$, be the resolvent of A and assume that $0 \in R(A)$. It is known that if E is a Hilbert space, then for each x in E, the strong $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$. This result was extended to a restricted class of Banach spaces in [3, 4]. In particular, Reich [6] showed that it is true under the assumption that E is a uniformly smooth Banach space, and that A is m-accretive. Rather unexpectedly, his proof involves the fixed point property for nonexpansive mappings.

In this paper, we establish a strong convergence theorem for resolvent $J_t x$ as $t \to \infty$ in a certain Banach space without using the fixed point property for nonexpansive mappings. Further we apply it to a new convergence result for an implicit iterative scheme.

2. Preliminaries

Let E be a real Banach space and let E^* its dual. $U=\{x\in E\colon \|x\|=1\}$ be unit sphere of E. Recall that the nom of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}(\|x+ty\|-\|x\|)/t$$

exists for each $x,y\in U$. It is said to be uniformly Gâteaux differentiable if for each y in U, this limit is attained uniformly as x varies over U. We shall write that E is (UG). The norm is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit is attained uniformly for $(x,y)\in U\times U$. Since E is uniformly smooth if and only if its dual E^* is uniformly convex, every Banach space with a uniformly convex dual is reflexive and (UG). But there are spaces E such that E is reflexive and (UG), but E is not even isomorphic to a uniformly smooth space [7, p. 149]. A discussion of these and related concepts may be found in [2].

The duality map from E into the family of nonempty subset of E^* is defined by

$$J(x)=\{x^*\in E^*\colon (x,x^*)=\|x\|^2=\|x^*\|^2\}$$

for each x in E, J is single-valued if and only if E is smooth.

Recall that a Banach limit LIM is a bounded linear functional on l^∞ of norm 1 such that

$$\liminf_{n\to\infty} t_n \leq \text{LIM } t_n \leq \limsup_{n\to\infty} t_n$$

and

$$\text{LIM } t_n = \text{LIM } t_{n+1}$$

for all $\{t_n\}$ in l^∞. Let $\{x_n\}$ be a bounded sequence in E. Then we can define the real valued continuous convex function ϕ on E by

$$\phi(E) = \text{LIM } \|x_n - z\|^2$$

for each $z \in E$.

By the method of [7], we obtain the following.

LEMMA 1. Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, let $\{x_n\}$ be a bounded sequence in E, and let LIM be a Banach limit. Let $u \in C$. Then

$$\text{LIM } \|x_n - u\|^2 = \inf \{\text{LIM } \|x_n - z\|^2 : z \in C\}$$

if and only if LIM $(z-u,\ J(x_n-u)) \leq 0$ for all $z \in C$.

PROOF. For z in C, and $0 \leq t \leq 1$, we have

$$\begin{aligned}\|x_n - u\|^2 &= \|x_n - tu - (1-t)z + (1-t)(z-u)\|^2 \\ &\geqq \|x_n - tu - (1-t)z\|^2 \\ &\quad + 2(1-t)(z-u,\ J(x_n - tu - (1-t))).\end{aligned}$$

Let $\varepsilon > 0$ be given. Since the norm of E is uniformly Gâteaux differentiable, the duality mapping is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^*. Therefore

$$||(z-u,\ J(x_n-tu-(1-t)z)-J(x_n-u))||<\varepsilon$$

if t is close enough to 1. Consequently, we have

$$(z-u,\ J(x_n-u))<\varepsilon+(z-u,\ J(x_n-tu-(1-t)z))$$
$$\leq \varepsilon+\frac{1}{2(1-t)}\{||x_n-u||^2-||x_n-tu-(1-t)z||^2\}$$

and hence

$$\text{LIM }(z-u,\ J(x_n-u))$$
$$\leq \varepsilon+\frac{1}{2(1-t)}\{\text{LIM }||x_n-u||^2-\text{LIM }||x_n-tu-(1-t)z||^2\}<\varepsilon.$$

Therefore, we have $\text{LIM}(z-u,\ J(x_n-u))\leq 0$ for all $z\in C$.

We prove the converse. Let $z,\ u\in C$. Then, since

$$||x_n-z||^2-||x_n-u||^2\geqq 2(u-z,\ J(x_n-u))$$

for all n and $\text{LIM }(z-u,\ J(x_n-u))\leq 0$, we have

$$\text{LIM }||x_n-u||^2=\inf\ \{\text{LIM }||x_n-z||^2 : z\in C\}.$$

REMARK. In Lemma 1, if $C=E$, then, for $u\in E$,

$$\text{LIM }||x_n-u||^2=\inf\{\text{LIM}||x_n-z||^2 : z\in E\}$$

if and only if $\text{LIM }(z,\ J(x_n-u))=0$ for all $z\in E$.

Let D be a subset of E. Then we denote the closure of D by $cl(D)$ and its distance from a point x in E by $d(x,D)$. We also denote the set $\{y\in D : ||y||=d(0,D)\}$ by D^0.

We conclude this section with the following lemma which is essentially well known. (cf. [1. p. 79].)

LEMMA 2. Let E be a Banach space and let C be a closed convex subset of E. If E is reflexive and strictly convex, then C^0 is a singleton.

3. Main results

Recall that an operator $A \subset E \times E$ is accretive if and only if for each $x_i \in D(A)$ and $y_i \in Ax_i, i = 1, 2,$ there exists $j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. If A is accretive, we can define, for each positive r, the resolvent of A, $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$ and the Yosida approximation of A, A_r: $R(I+rA) \to E$ by $A_r = \frac{1}{r}(I - J_r)$. We know that $A_r x \in AJ_r x$ for every $x \in R(I+rA)$ and that $\|A_r x\| \leq |Ax|$ for every $x \in D(A) \cap R(I+rA)$, where $|Ax| = \inf\{\|y\| : y \in Ax\}$. We also know that $A^{-1}0 = F(J_r)$ for each $r > 0$, where $F(J_r)$ is the set of fixed points of J_r.

LEMMA 3. Let E be a Banach space, let $A \subset E \times E$ be an accretive operator that satisfies the range condition: $R(I+rA) \supset cl(D(A))$ for all $r > 0$.

(I) If there exists $\{t_n\}$ with $t_n \to \infty$ and $y = \lim_{n\to\infty} J_{t_n} x$, then $y \in A^{-1}0$.

(II) If E is smooth and there exist $\{t_n\}$ and $\{s_n\}$ such that $t_n \to \infty$, $y = \lim_{n\to\infty} J_{t_n} x$ and $z = \lim_{n\to\infty} J_{s_n} x$, then $y = z$.

PROOF. (I) Let $r > 0$. Since $y = \lim_{n\to\infty} J_{t_n}$ and hence $\{J_{t_n} x\}$ is bounded, we have

$$\|J_r J_{t_n} x - J_{t_n} x\| \leq r|AJ_{t_n} x| \leq r\|(x - J_{t_n} x)/t_n\| \to 0,$$

as $n \to \infty$. Then we have $J_r y = y$ and hence $y \in A^{-1}0 = F(J_r)$.

(II) Since $z \in A^{-1}0$ and A is an accretive operator, we have

$$(J_{t_n}x - x,\ J(J_{t_n}x - z)) \leq 0$$

and hence $(y-x,\ J(y-z)) \leq 0$. Similarly, we have $(z-x, J(z-y)) \leq 0$. Therefore $(y-z, J(y-z)) \leq 0$, that is, $y=z$.

Now we establish the behavior of $J_t x$ as $t \to \infty$.

THEOREM 1. Let E be a reflexive and strictly convex Banach space, and let $A \subset E \times E$ be an accretive operator that satisfies the range condition. Let C be a closed convex subset of E such that $cl(D(A)) \subset C \subset \bigcap_{r>0} R(I+rA)$. If E is (UG) and $0 \in R(A)$, then for each x in C, $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$.

PROOF. Fix a point x in C and a positive r. Let $t_n \to \infty$, $x_n = J_{t_n}x$ and $y_n = (x - x_n)/t_n$. Then, since $A^{-1} \neq \phi$, $\{x_n\}$ is bounded. So for a Banach limit LIM, we can define a real valued function ϕ on C by

$$\phi(z) = \mathrm{LIM}\|x_n - z\|^2$$

for each $z \in C$. Since ϕ is continuous, convex and $\phi(z) \to \infty$ as $\|z\| \to \infty$ while E is reflexive, it attains its infimum over C. Let

$$K = \{u \in C : \phi(u) = \inf\{\phi(z) : z \in C\}\}.$$

Then it follows that K is nonempty, closed, convex and bounded. Furthermore, K is invariant under J_r. In fact, since $\|J_r x_n - x_n\| \to 0$ as $n \to \infty$, we have for each $u \in K$,

$$\begin{aligned}\phi(J_r u) &= \mathrm{LIM}\,\|x_n - J_r u\|^2 \\ &= \mathrm{LIM}\,\|J_r x_n - J_r u\|^2 \\ &\leq \mathrm{LIM}\,\|x_n - u\|^2 = \phi(u).\end{aligned}$$

We also observe that K contains a fixed point of J_r. To see this, let $w \in A^{-1}0$ and define

$$K' = \{u \in K : \|u - w\| = d(w, K)\}.$$

By Lemma 2, K' is a singleton. Denote such a singleton by v. Then $\|J_r v - w\| = \|J_r v - J_r w\| \leq \|v - w\|$, so that $J_r v = v$. Since $v \in A^{-1}0$ and A is accretive, we have, on the one hand, $(x_n - x,\ J(x_n - v)) \leq 0$ for all n and hence

$$\mathrm{LIM}(x_n - x,\ J(x_n - v)) \leq 0. \tag{1}$$

Since $v \in K$, by Lemma 1, we have, on the other hand,

$$\mathrm{LIM}(z - v,\ J(x_n - v)) \leq 0$$

for all $z \in C$. Putting $z = x$, we have

$$\mathrm{LIM}(x - v,\ J(x_n - v)) \leq 0. \tag{2}$$

Combining (1) and (2), we have $\mathrm{LIM}\|x_n - v\|^2 \leq 0$. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to v. Therefore, by Lemma 3, we obtain that $\lim_{t\to\infty} J_t x = v$.

COROLLARY 1. Let E be a reflexive and strictly convex Banach space. Let $A \subset E \times E$ be an accretive operator that satisfies the range condition and $0 \in R(A)$. If E is (UG) and $cl(D(A))$ is convex, then for each $x \in cl(D(A))$, $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$.

PROOF. Putting $C = cl(D(A))$, we can obtain the desired result.

COROLLARY 2. Let E be a reflexive and strictly convex and let $A \subset E \times E$ be an m-accretive operator. If E is (UG) and $0 \in R(A)$, then for each $x \in E$, $\lim_{t\to\infty} J_t x$ exists and belongs to $A^{-1}0$.

PROOF. Putting $C = E$, we can obtain the desired result.

4. Application

We consider the implicit iterative scheme

$$x_{n+1} - x_n + h_{n+1}(y_{n+1} + p_{n+1}(x_{n+1} - z)) = w_{n+1},\ n \geq 0 \qquad (3)$$

where $z \in E$, $y_n \in Ax_n$, $\sum_{n=1}^{\infty} |w_n| < \infty$, and $\{h_n\}$ and $\{p_n\}$ are positive sequences such that $\{p_n\}$ decreases to 0, $\{h_n p_n\}$ is bounded, $\sum_{n=1}^{\infty} h_n p_n = \infty$ and $\lim_{n\to\infty} (p_{n-1}/p_n - 1)/p_n h_n = 0$.

Corollary 2 implies that [5. Theorem] is valid in all Banach spaces, which are (UG), reflexive and strictly convex.

THEOREM 2. Let E be a reflexive and strictly convex Banach space, and let $A \subset E \times E$ be m-accretive. If E is (UG) and $0 \in R(A)$, then sequence $\{x_n\}$ defined by (3) converges strongly to a zero of A.

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