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SOME REMARKS ON THE CARTAN-THULLEN THEOREM FOR LOCALLY CONVEX SPACES

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1. Introduction

Let E be a complex locally convex space and U be an open subset of E. CS(E) denotes the set of all continuous seminorms on E. For $\xi\in E$, $r>0$ and $\alpha\in$CS(E) let $B_{\alpha,r}(\xi)=\{x\in E\colon\ \alpha(x-\xi)<r\}$.

Let $H(U)$ be the vector space of all complex valued holomorphic functions on U. The Nachbin topology and the Coeuré-Nachbin topology on $H(U)$ are denoted by τ_ω and τ_δ, respectively. (see [1].)

When E is an infinite dimensional complex Banach space, the Cartan-Thullen theorem was studied by Dineen [2, 3], Gruman and Kiselman [4], Matos [5] and so on. Specially, Dineen [3] showed the Cartan-Thullen theorem under the condition $\tau_\omega=\tau_\delta$ on $H(U)$. In this article we show that the Cartan-Thullen theorem holds for a complex locally convex space under the condition $\tau_\omega=\tau_\delta$ on $H(U)$.

2. Main results

DEFINITION 1. An open subset U of E is said to be a *domain of holomorphy* if there do not exist two nonvoid connected open subsets V and W in E such that

(1) $W \subset U \cap V$ and V is not contained in U.
(2) For every $f \in H(U)$ there exists $\tilde{f} \in H(V)$ such that $f = \tilde{f}$ on W.

DEFINITION 2. An open subset U of E is said to be the *domain of existence* of a function $f \in H(U)$ if there do not exist two nonvoid connected open subsets V and W in E and $\tilde{f} \in H(V)$ such that

(1) $W \subset U \cap V$ and V is not contained in U.
(2) $f = \tilde{f}$ on W.

It is clear that every domain of existence is a domain of holomorphy.

DEFINITION 3. Let U be an open subset of E.

(1) The *holomorphic convex hull* of a set $A \subset U$ is defined by

$$\hat{A} = \{x \in U : |f(x)| \leq \sup_{x \in A} |f(x)| \text{ for all } f \in H(U)\}.$$

(2) The open set U is said to be *holomorphically convex* if for each compact set $K \subset U$ there exists $\alpha \in \mathrm{CS}(E)$ such that $\alpha(\hat{K}, \partial U) \neq 0$ where ∂U denotes the set of all boundary points of U.

THEOREM 1. Let U be an open subset of E. If for every net $\{x_\lambda\}_{\lambda \in I}$ in U converging to a boundary point $\xi \in \partial U$, there exists $f \in H(U)$ such that $\sup_{\lambda \in I} |f(x_\lambda)| = \infty$, then U is a domain of holomorphy.

PROOF. Suppose that U is not a domain of holomorphy. Then there exist two nonvoid connected open subsets V and W such that (1) $W \subset U \cap V$ and V is not contained in U (2) For every $f \in H(U)$ there exists $\tilde{f} \in H(V)$ with $f = \tilde{f}$ on W. Without loss of generosity, we may assume that W is a connected component of $U \cap V$. Then there exists $\xi \in \partial U \cap V \cap \partial W$ which is a limit of some net $\{x_\lambda\}_{\lambda \in I}$ in W. By the hypothesis, there exists $f \in H(U)$ such that $\sup_{\lambda \in I} |f(x_\lambda)| = \infty$. However, $\lim \tilde{f}(x_\lambda) = \tilde{f}(\xi)$ and this leads to contradiction because $f = \tilde{f}$ on W.

THEOREM 2. Every domain of holomorphy is holomorphically convex.

PROOF. Suppose that an open subset U of E is a domain of holomorphy. Let K be a compact subset of U. Then there exists $\alpha \in \mathrm{CS}(E)$ such that $\alpha(K, \partial U) > 0$ and set $r = \alpha(K, \partial U)$. Let $\xi \in \hat{K}$ and $f \in H(U)$. Given $t \in B_{\alpha, r}(0)$, there exists $\rho > 1$ such that $\rho t \in B_{\alpha, r}(0)$.

Then $K + \{\lambda \rho t : |\lambda| \leq 1\}$ is a compact subset of U, and so we can find an open neighborhood W of 0 such that

$$B = K + \{\lambda \rho t : |\lambda| \leq 1\} + \rho W \subset U$$

and

$$M = \sup_{x \in B} |f(x)| < \infty.$$

If $h \in W$, then we have

$$|\hat{d}^m f(\xi)\ (t+h)| \leq \sup_{x\in K} |\hat{d}^m f(x)(t+h)|$$
$$\leq \rho^{-m} M$$

by the Cauchy inequality and the fact $\xi \in \hat{K}$.

Thus for each $t \in B_{\alpha,r}(0)$, there exists an open neighborhood W of 0 such that the series $\sum_{m=0}^{\infty} \hat{d}^m f(\xi)(t+h)$ converges uniformly for all $h \in W$. This shows that

$$f_\xi(x) = \sum_{m=0}^{\infty} \hat{d}^m f(\xi)(x-\xi) \quad (x \in B_{\alpha,r}(\xi))$$

is a holomorphic function on $B_{\alpha,r}(\xi)$ and that $f_\xi = f$ on some neighborhood of ξ contained in $B_{\alpha,r}(\xi) \cap U$. Thus $B_{\alpha,r}(\xi) \subset U$ because U is a domain of holomorphy. This shows $\alpha(\hat{K}, \partial U) \geq r$ and hence U is holomorphically convex.

THEOREM 3. If U is holomorphically convex and $\tau_\omega = \tau_\delta$ on $H(U)$, for every net $\{x_\lambda\}_{\lambda\in I}$ in U converging to a boundary point $\xi \in \partial U$, there exists $f \in H(U)$ such that $\sup_{\lambda\in I} |f(x_\lambda)| = \infty$.

PROOF. Assume that $\tau_\omega = \tau_\delta$ on $H(U)$. Suppose that there exists a net $\{x_\lambda\}_{\lambda\in I}$ in U converging to a boundary point $\xi \in \partial U$ such that $\sup_{\lambda\in I} |f(x_\lambda)| < \infty$ for every $f \in H(U)$. Then the set $B = \{x_\lambda : \lambda \in I\}$ is a bounding subset of U and so the seminorm $p(f) = \sup_{x\in B} |f(x)|$ $(f \in H(U))$ is τ_δ continuous, hence τ_ω continuous on $H(U)$. Thus there exists a compact subset K of U such that to every open subset V, $K \subset V \subset U$, there corresponds a constant $C(V) > 0$ with

$$p(f) \leq C(V) \sup_{x \in V} |f(x)| \quad (f \in H(U)).$$

Replacing f with f^n, taking the n-th root, and letting $n \to \infty$, we obtain

$$p(f) \leq \sup_{x \in V} |f(x)| \quad (f \in H(U)).$$

Since this holds for every open set $V \supset K$, we have

$$p(f) \leq \sup_{x \in K} |f(x)| \quad (f \in H(U)).$$

This means that for every $\lambda \in I$,

$$|f(x_\lambda) \leq \sup_{x \in K} |f(x)| \quad (f \in H(U))$$

and so $B \subset \hat{K}$. Hence we have $\xi \in \partial\hat{K} \cap \partial U$ and $\alpha(\hat{K}, \partial U) = 0$ for every $\alpha \in \mathrm{CS}(E)$. This contradicts the fact that U is holomorphically convex.

THEOREM 4. (Cartan-Thullen) Let U be an open subset of E. Consider the following statements.

(a) For every net $\{x_\lambda\}_{\lambda \in I}$ in U converging to a boundary point $\xi \in \partial U$, there exists $f \in H(U)$ such that $\sup_{\lambda \in I} |f(x_\lambda)| = \infty$.

(b) U is a domain of existence.

(c) U is a domain of holomorphy.

(d) U is holomorphically convex.

Then the following implications hold:

$$(a) \to (c) \quad \text{and} \quad (b) \to (c) \to (d).$$

If $\tau_\omega = \tau_\delta$ on $H(U)$, then the statements (a), (c) and (d) are equivalent.

PROOF. This follows immediately from Theorem 1, Theorem 2 and Theorem 3.

References

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