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ON CERTAIN INTEGRALS OF ANALYTIC FUNCTION

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Abstract

The object of the present paper is to derive some inequalities for certain integrals of functions belonging to the classes $A(n)$, $S^*(n;\alpha)$ and $K(n,\alpha)$. As the special class of our theorems, we have the corresponding result shown by M. Obradovič [2].

1. Introduction

Let $A(n)$ be the class of functions of the form

$$(1.1)\quad f(z)=z+\sum_{n+1}^{\infty} a_k z^k \quad (n\in N=\{1,2,3,\cdots\})$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$.

A function $f(z)$ belonging to $A(n)$ is said to be *in the class* $S^*(n,\alpha)$ if and only if

$$(1.2)\quad Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for some $\alpha(0\leqq\alpha<1)$ and for all $z\in U$.

A function $f(z)$ belonging to $A(n)$ is said to be *in the* *class* $K(n, \alpha)$ if and only if

$$(1.3) \quad Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha$$

for some $\alpha(0 \leqq \alpha < 1)$ and for all $z \in U$. We note that $f(z) \in K(n, \alpha)$ if and only if $zf'(z) \in S^*(n, \alpha)$ for $0 \leqq \alpha < 1$, and that $K(n, \alpha) \subset S^*(n, \alpha)$ for $0 \leqq \alpha < 1$.

In the present paper, we prove some inequalities for certain integrals of functions $f(z)$ belonging to the classes $A(n)$, $S^*(n, \alpha)$ and $K(n, \alpha)$.

2. Inequalities for certain integrals

We begin with the statement of the following lemma due to Miller and Mocanu[1].

LEMMA. Let $\phi(u, v)$ be a complex valued function,

$$\phi: D \longrightarrow C, \quad D \subset C \times C \ (C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies the following conditions:

(i) $\phi(u, v)$ is continuous in D;

(ii) $(1, 0) \in D$ and $Re\{\phi(1, 0)\} > 0$;

(iii) $Re\{\phi(iu_2, v_1)\} \leqq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leqq -n(1+u_2^2)/2$.

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots\cdots$ be regular in the open unit disk U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\phi(p(z), zp'(z))\} > 0 \quad (z \in U),$$

then $Re\{p(z)\} > 0 \quad (z \in U)$.

Now, we prove

THEOREM 1. Let the function $f(z)$ defined by (1.1) be in the class $A(n)$, $\alpha<1$ and $a>-1$. If $Re\{f(z)/z\}>\alpha$, then

$$(2.1)\quad Re\left\{\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt\right\}>\alpha+\frac{n(1-\alpha)}{2(a+1)+n} \text{ for } z\in U.$$

PROOF. We define the function $p(z)$ by

$$(2.2)\quad \frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt=\beta+(1-\beta)p(z),$$

where

$$(2.3)\quad \beta=\alpha+\frac{n(1-\alpha)}{2(a+1)+n}.$$

Then the function $p(z)=1+p_n z^n+p_{n+1}z^{n+1}+\cdots$ is regular in U. It follows from (2.2) that

$$(2.4)\quad \int_0^z t^{a-1}f(t)dt=\frac{z^{a+1}}{a+1}(\beta+(1-\beta)p(z)).$$

Further, taking the differentiations of both sides in (2.4), we see that

$$(2.5)\quad \frac{f(z)}{z}=\beta+(1-\beta)p(z)+\frac{1-\beta}{a+1}zp'(z),$$

that is, that

$$(2.6)\quad Re\left\{\frac{f(z)}{z}-\alpha\right\}=Re\{\beta-\alpha+(1-\beta)p(z)$$
$$+\frac{1-\beta}{a+1}zp'(z)\}>0 \text{ for } z\in U.$$

Let $p(z)=u=u_1+iu_2$, $zp'(z)=v=v_1+iv_2$, and

$$(2.7)\quad \phi(u,v)=\beta-\alpha+(1-\beta)u+\frac{1-\beta}{a+1}v.$$

Then the function $\phi(u,v)$ satisfies;

(i) $\phi(u,v)$ is continuous in $D=C\times C$,

(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}=1-\alpha>0$,

(iii) for all $(iu_2,\ v_1)\in D$ such that $v_1\leqq -n(1+u_2^2)/2$,

$$\begin{aligned} Re\{\phi(iu_2,\ v_1)\} &= \beta-\alpha+\frac{1-\beta}{a+1}v_1 \\ &\leqq \beta-\alpha-\frac{n(1-\beta)(1+u_2^2)}{2(a+1)} \\ &\leqq 0 \end{aligned}$$

for given by (2.3). Therefore, the function $\phi(u,v)$ satisfies the condition in Lemma.

Applying Lemma, we have $Re\{p(z)\}>0$ $(z\in U)$, or

$$Re\left\{\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt\right\}>\beta\quad (z\in U).$$

This completes the assertion of Theorem 1.

REMARK 1. Letting $n=1$ in Theorem 1, we have the corresponding result which was proved by Obradović [2].

Making $zf'(z)$ instead of $f(z)$, Theorem 1 gives

COROLLARY 1. Let the function $f(z)$ defined by (1.1) be in the class $A(n)$, $\alpha<1$ and $a>-1$. If $Re\{f'(z)\}>\alpha$, then

$$(2.8)\quad Re\left\{\frac{a+1}{z^{a+1}}\int_0^z t^a f'(t)dt\right\}>\alpha+\frac{n(1-\alpha)}{2(a+1)+n}\ \text{ for }\ z\in U.$$

Next, we derive

THEOREM 2. Let the function $f(z)$ defined by (1.1) be in the class $S^*(n,\alpha)$ and $a>-1$. Then we have

$$(2.9)\quad Re\left\{\frac{z^a f(z)}{\int_0^z t^{a-1}f(t)dt}\right\}>\beta \quad (z\in U),$$

where

$$(2.10)\quad \beta=\frac{2a+2\alpha-1+\sqrt{(2a+2\alpha-1)^2+8n(a+1)}}{4}.$$

PROOF. Define the function $p(z)$ by

$$(2.11)\quad \frac{z^a f(z)}{(a+1)\int_0^z t^{a-1}f(t)dt}=\gamma+(1-\gamma)p(z),$$

where $\gamma=\beta/(a+1)$. Then $p(z)$ is regular in U and $p(z)=1+p_n z^n+p_{n+1}z^{n+1}+\cdots$. Making the logarithmic differentiation of both sides in (2.11), we have

$$(2.12)\quad \frac{zf'(z)}{f(z)}=(a+1)\gamma-a+(a+1)(1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)},$$

or

$$(2.13)\quad Re\left\{(a+1)\gamma-a-\alpha+(a+1)(1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\right\}>0.$$

Letting $p(z)=u=u_1+iu_2$, $zp'(z)=v=v_1+iv_2$, and

$$(2.14)\quad \phi(u,v)=(a+1)\gamma-a-\alpha+(a+1)(1-\gamma)u + \frac{(1-\gamma)v}{\gamma+(1-\gamma)u},$$

we see that

(i) $\phi(u,v)$ is continuous in $D=\left(C-\left\{\frac{\gamma}{\gamma-1}\right\}\right)\times C$,

(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}=1-\alpha>0$,

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\leqq -n(1+u_2^2)/2$,

$$\begin{aligned} Re\{\phi(iu_2,\ v_1)\} &= (a+1)\gamma-a-\alpha+\frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2} \\ &\leqq (a+1)\gamma-a-\alpha-\frac{n\gamma(1-\gamma)(1+u_2^2)}{2\{\gamma^2+(1-\gamma)^2u_2^2\}} \\ &\leqq 0. \end{aligned}$$

Since the function $\phi(u,v)$ satisfies the conditions in Lemma, we conclude that

$$(2.14)\quad Re\left\{\frac{z^a f(z)}{(a+1)\int_0^z t^{a-1}f(t)dt}\right\}>\gamma \quad (z\in U),$$

which implies (2.9).

COROLLARY 2. If the function $f(z)$ defined by (1.1) is in the class $K(n,\alpha)$, then

$$(2.15)\quad Re\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{-2\alpha-1+\sqrt{(2\alpha-1)^2+8n}}{4} \quad (z\in U).$$

PROOF. Note that $f(z)\in K(n,\alpha)$ if and only if $zf'(z)\in S^*(n,\alpha)$. Therefore, letting $zf'(z)$ instead of $f(z)$ and $a=0$ in Theorem 2, we have the assertion of Corollary 2.

Finally, we prove

THEOREM 3. Let the function $f(z)$ defined by (1.1) be in the class $K(n,\alpha)$ and $a>-1$. Then

$$(2.16)\quad Re\left\{\frac{zf'(z)}{f(z)-\frac{a}{z^a}\int_0^z t^{a-1}f(t)dt}\right\}>\beta \quad (z\in U),$$

where β is given in (2.10).

PROOF. Noting that $f(z) \in K(n, \alpha)$ if and only if $zf'(z) \in S^*(n, \alpha)$, and using Theorem 2, we easily show the inequality (2.16).

REMARK 2. Letting $a=0$ in Theorem 3, we also have the same statement of Corollary 2.

References

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