Pusan Kyŏngnam Mathematical
Vol. 4, 1~13, 1988

ON SOME PROPERTIES OF BOUNDED HOMOMORPHISMS AND DERIVATIONS OF A C^*-ALGEBRA

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We consider some properties of the completely bounded representations of C*-algebras. We discuss the relation between the k-similarity and the property D_k and get the result every k-similar C^*-algebra has property D_k. Moreover we determine the similarity problem for the algebra $C \oplus C$ precisely and constructively.

1. Introduction

Let ϕ be a bounded non-degenerate representation of a C^*-algebra A on a Hilbert space H. The following fact is well known:

ϕ is similar to a $*$-representation if and only if φ is completely bounded.

Haagerup ([5]) has shown that there exists a bounded invertible operator T on H such that $T\phi T^{-1}$ is a $*$-representation and $\|T\| \, \|T^{-1}\| = \|\phi\|_{cb}$. Recently Christensen ([4]) has shown that any representation ϕ of a II_1-factor with property Γ is completely bounded and $\|\phi\|_{cb} \leqq \|\phi\|^{44}$.

Therefore to determine a similarness of a bounded representation will be assigned to determine its complete boundedness. So it is interesting to estimate a complete bounded norm of a bounded representation and to study the connection between this norm and its original norm.

According to these motivations we will define the following notations.

DEFINITION 1. Let k be a positive real and ϕ be a bounded linear map from a C^*-algebra A into a C^*-algebra B. Then ϕ is said to be *k-completely bounded*, if ϕ is completely bounded and $\|\phi\|_{cb} \leqq \|\phi\|^k$.

For example, each completely positive map ϕ is completely bounded and $\|\phi\|_{cb} = \|\phi\|$, and every cyclic bounded representation is 3-completely bounded. (cf. [5].)

DEFINITION 2. A C^*-algebra A is said to be *k-similar*, if every bounded non-degenerate representation of A on a Hilbert space H is k-completely bounded.

As mentioned above we will see that for any k-similar C^*-algebra A, every bounded representation of A is similar to a $*$-representation. The following fact is known:

A nuclear C^*-algebra is 2-similar (cf. [1], [2]).

A C^*-algebra which has no tracial states is 3-similar.

A properly infinite von Neumann algebra is 3-similar ([5]), and a type II_1-factor with property Γ is 44-similar.

But these estimations are not necessarily best possible. In fact we can show that a 2-dimensional C^*-algebra $C\oplus C$ is 1-similar.

2. k-similarity

In this section, we will discuss some preperties of k-similar C^*-algebras. At first we note the following fact.

LEMMA 3. Let x and 1 (the identity operator) be in $B(H)$ and λ be a positive real. Then we have a norm of an operator $\begin{pmatrix}1 & x\\ 0 & \lambda\end{pmatrix}$ in $B(H\oplus H)$,

$$\left\|\begin{pmatrix}1 & x\\ 0 & \lambda\end{pmatrix}\right\|=\frac{1}{2}[\{\|x\|^2+(\lambda+1)^2\}^{1/2}+\{\|x\|^2+(\lambda-1)^2\}^{1/2}].$$

PROOF. Let $x=u|x|$ be a polar decomposition of x. We compute the norm of an operator $\begin{pmatrix}1 & x\\ 0 & \lambda\end{pmatrix}$, so we may assume that this partial isometry u is a unitary operator from H to H. Using this unitary operator u, we can identify the algebra $B(H\oplus H)$ with $B(H)\otimes M_2$. That is, this identification is as follows;

$$\begin{array}{ccc} B(H\oplus H) & & B(H)\otimes M_2 \\ \Cup & & \Cup \\ uau^*+ub+cu^*+d & \longrightarrow & \begin{pmatrix}a & b\\ c & d\end{pmatrix}\end{array}$$

Therefore, we can calculate the norm of $\begin{pmatrix}1 & |x|\\ 0 & \lambda\end{pmatrix}$ in $B(H)\otimes M_2$. Let $|x|=\int_0^{\|x\|}\mu\, de(\mu)$ be the spectral decomposition of $|x|$. Since we get $\begin{pmatrix}1 & |x|\\ 0 & \lambda\end{pmatrix}=\int\begin{pmatrix}1 & \mu\\ 0 & \lambda\end{pmatrix}dE(\mu)$, where $dE(\mu)=\begin{pmatrix}de(\mu) & 0\\ 0 & de(\mu)\end{pmatrix}$, then we have

$$\left\|\begin{pmatrix} 1 & \|x\| \\ 0 & \lambda \end{pmatrix}\right\| = \sup\left\{\left\|\begin{pmatrix} 1 & \mu \\ 0 & \lambda \end{pmatrix}\right\| : 0 \leqq \mu \leqq \|x\|\right\}$$
$$= \left\|\begin{pmatrix} 1 & \|x\| \\ 0 & \lambda \end{pmatrix}\right\|$$
$$= \frac{1}{2}[\{\|x\|^2 + (\lambda+1)^2\}^{1/2} + \{\|x\|^2 + (\lambda-1)^2\}^{1/2}].$$

The proof is complete.

Let e be an idempotent operator on H and p be the range projection of e. Then $(1-p)e=0$. Therefore we can represent e as following form

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \in B(pH \oplus (1-p)H),$$

where x is an operator from $(1-p)H$ to pH.

COROLLARY 4. If $\{\phi, H\}$ is a bounded representation of a C^*-algebra A, then the following two conditions are equivalent:

(1) $\|\phi\|=1$.

(2) ϕ is $*$-representation.

PROOF. (2) implies (1): It is obvious. (1) implies (2): By considering the transposed map of the restriction of ${}^t\phi$ to $B(H)^*$, we may assume the ϕ is a normal representation of a von Neumann algebra A^{**} on H. Hence we have only to show that $\phi(p)$ is a projection for any projection p in A^{**}.

We assume that $\phi(p)$ is not projection. Then $\phi(p)$ is an idempotent, so $\phi(p)$ has the form $\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$ as mentioned above, where $x \neq 0$. However, we have $\|\phi(p)\| > 1$ by Lemma 3, which contradict to $\|\phi\| = 1$.

From now on we will consider the k-similarity of a C^*-algebra $C\oplus C$. Let $\{\phi, H\}$ be a representation of the 2-dimensional algebra $C\oplus C$. If we denote by H_1 and H_2 the range space of the idempotent $\phi(1,0)$ and its orthogonal complement respectively, then we can represent the operator $\phi(1,0)$ by the form $\begin{pmatrix}1 & x\\ 0 & 0\end{pmatrix}\in B(H\oplus H)=B(H)$, and $\phi(0,1)$ has the form $\begin{pmatrix}0 & -x\\ 0 & 1\end{pmatrix}$. We define a bounded invertible operator S and a $*$-representation π of $C+C$ on H as follows

$$S=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\in B(H_1\oplus H_2)$$

and

$$\pi(\alpha,\ \beta)=\alpha p_1\oplus\beta p_2,$$

where p_1 is a projection from H to H_1. Then we can see that $\phi=S^{-1}\pi S$, that is, ϕ is similar to this $*$-representation π. Moreover we consider the bounded invertible operator

$$T=\begin{pmatrix}1 & 0\\ 0 & (||x||^2+1)^{1/2}\end{pmatrix}S.$$

Then we get the following result.

THEOREM 5. For any bounded non-degenerate representation ϕ of $C\oplus C$ on a Hilbert space H, we use the above notations. Then

(1) $||\phi||=||\phi(1,-1)||$. (the norm of a difference of two idempotents.)

(2) $\phi=T^{-1}\pi T$.

(3) $||T||\ ||T^{-1}||=||\phi||$.

In particular, $C\oplus C$ is 1-similar.

PROOF. (1) For any $\alpha,\beta\in C$, we can represent $\phi(\alpha,\beta)$ by the following from

$$\phi(\alpha, \beta)=\begin{pmatrix}\alpha & (\alpha-\beta)x \\ 0 & \beta\end{pmatrix}.$$

We compute this norm. We may assume $|\alpha|=|\beta|=1$, so

$$||\phi|| \leqq \left\|\begin{pmatrix}1 & 2||x|| \\ 0 & 1\end{pmatrix}\right\|=||x||+(||x||^2+1)^{1/2}.$$

Conversely,

$$||\phi(1, -1)||=\left\|\begin{pmatrix}1 & 2x \\ 0 & 1\end{pmatrix}\right\|=||x||+(||x||^2+1)^{1/2}.$$

Thus we get

$$||\phi||=||\phi(1, -1)||.$$

(2) It is clear that we note the fact $\begin{pmatrix}1 & 0 \\ 0 & (||x||^2+1)^{1/2}\end{pmatrix}$ belongs to the center of the range of π.

(3) Let $\lambda=(||x||^2+1)^{1/2}$. We note that T and T^{-1} have the following form,

$$T=\begin{pmatrix}1 & 0 \\ 0 & \lambda\end{pmatrix}\begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x \\ 0 & \lambda\end{pmatrix},$$

$$T^{-1}=\begin{pmatrix}1 & -x \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & 1/\lambda\end{pmatrix}=\begin{pmatrix}1 & -x/\lambda \\ 0 & 1/\lambda\end{pmatrix}.$$

Using Lemma 3, we can compute norms of $||T||$, $||T^{-1}||$ and get $||T||\,||T^{-1}||=||x||+(||x||^2+1)^{1/2}$. The proof is complete.

PROPOSITION 6. Let A and B be C^*-algebras and k be a positive real. Then the following holds:

(1) If $A \oplus B$ is k-similar, then A and B are k-similar.

(2) If both A and B are k-similar, then $A \oplus B$ is $(k+1)$-similar.

(3) Suppose that $\{A_n\}$ is an increasing sequence of k_n-

similar C^*-subalgebras of A, $A=\bigcup \bar{A}_n$ and $N=\sup k<\infty$, then the C^*-algebra A is N-similar.

PROOF. (1) It easily follows by identifying A and B with $A+0$ and $0+B$ in $A+B$ respectively.

(2) Let ϕ be a bounded non-degenerate representation of $A+B$ on a Hilbert space H. Put $\tilde{\phi}=\phi|_{C\oplus C}$. By Theorem 5 there exists a bounded invertible operator T on H such that $T\phi T^{-1}$ is a $*$-representation of $C\oplus C$ on H and $\|\tilde{\phi}\|=\|\tilde{\phi}\|_{cb}$. Hence we denote by p_1, p_2, orthogonal projections $T\phi(1,0)T^{-1}$, $T\phi(0,1)T^{-1}$ with sum 1 respectively. Putting $H_i=p_iH$, $\phi_i=T\phi T^{-1}|_{H_i}(i=1,2)$, it follows that ϕ_1 and ϕ_2 are bounded non-degenerate representations of A and B on H_1 and H_2 respectively, and $T\phi T^{-1}=\phi_1\oplus\phi_2$. Since T is of the form $\begin{pmatrix} 1 & x \\ 0 & \lambda \end{pmatrix}$ by Theorem 5 for each $a\in A$ and $b\in B$, we have

$$\begin{aligned}\phi(a,b)&=T^{-1}(\phi_1(a)\oplus\phi_2(b))T\\&=\begin{pmatrix} 1 & -x/\lambda \\ 0 & 1/\lambda \end{pmatrix}\begin{pmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(b) \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & \lambda \end{pmatrix}\\&=\begin{pmatrix} \phi_1(a) & \phi_1(a)x-x\phi_2(b) \\ 0 & \phi_2(b) \end{pmatrix}.\end{aligned}$$

Hence $\|\phi\|\geqq\max\{\|\phi_1\|,\|\phi_2\|\}=\|\phi_1\oplus\phi_2\|$. Noting that $\phi_1\oplus\phi_2$ is k-completely bounded because ϕ_1 and ϕ_2 are k-completely bounded by assumption, we have

$$\begin{aligned}\|\phi\|_{cb}&=\|T^{-1}(\phi_1\oplus\phi_2)T\|_{cb}\\&\leqq\|T\|\ \|T^{-1}\|\ \|\phi_1\oplus\phi_2\|_{cb}\\&\leqq\|\phi\|\ \|\phi_1\oplus\phi_2\|^{k'}\\&\leqq\|\phi\|\ \|\phi\|^k=\|\phi\|^{k+1}.\end{aligned}$$

Therefore ϕ is $(k+1)$-completely bounded.

(3) Suppose that ϕ is a bounded non-degenerate representation of $\cup \bar{A}_n$ on a Hilbert space H. We put $\phi_n = \phi | A_n$ for each $n \in N$, then ϕ_n is a bounded representation of A_r. Then we can easily show that $||\phi|| = \sup ||\phi_n||$, therefore the assertion follows.

COROLLARY 7. Let A be a C^*-algebra and I be a closed ideal of A. If A is k-similar, then I and A/I(the qoutient algebra) are k-similar. Conversely, if I and A/I are k-similar, then A is $(k+1)$-similar.

PROOF. Suppose that ϕ is a bounded non-degenerate representation of I on a Hilbert space H. Then it is extended to a normal representation $\tilde{\phi}$ of I^{**} on H with $||\phi|| = ||\tilde{\phi}||$. Since I^{**} is a σ-weakly closed ideal of a von Neumann algebra A^{**}, I^{**} is of form $A^{**}z$ with a central projection z of A^{**}. Put

$$\psi(x) = \tilde{\phi}(xz), \qquad x \in A^{**}.$$

Then ψ is a normal representation of A^{**} on H such that $\tilde{\phi} = \psi|_{I^{**}}$ and $||\psi|| = ||\tilde{\phi}|| (= ||\phi||)$. We note that if A is k-similar, then every normal representation of A^{**} is k-completely bounded. Thus we have

$$||\phi||_{cb} = ||\tilde{\phi}||_{cb} = ||\psi||_{cb'} \leqq ||\psi||^k = ||\tilde{\phi}||^k = ||\phi||^k.$$

Next, suppose that ϕ is a bounded non-degenerate representation of A/I on H and q is the quotient map from A onto A/I. Put $\tilde{\phi} = \phi \circ q$. Then $\tilde{\phi}$ is a bounded non-degenerate representation of A on H. By assumption there exists a bounded invertible operator S on H such that $\rho = S\tilde{\phi}S^{-1}$ is a $*$-representation of A on H and $||S||\ ||S^{-1}|| = ||\tilde{\phi}||_{cb}$.

We define a $*$-representation of A/I on H by

$$\pi(q(a))=\rho(a)$$

for each $a\in A$. Then it follows that $\pi=S\phi S^{-1}$ and

$$\|\phi\|_{cb}\leqq\|S\|\ \|S^{-1}\|=\|\tilde{\phi}\|_{cb}\leqq\|\tilde{\phi}\|^{k}\leqq\|\phi\|^{k}.$$

The converse assertion follows from the fact that the algebra A^{**} can be identified a direct sum of I^{**} and $(A/I)^{**}$ and Proposition 6.

3. The relation to property D_k

As stated in introduction, the similarity problems concerned with the derivation problem. Specially, we consider here the relation between k-similarity and property D_k. We recall the following definition.

DEFINITION 8. A C^*-algebra A is said to have property D_k for some positive real k, if for each non-degenerate $*$-representation ϕ of A on a Hilbert space H we have

$$d(x,\phi(A)')\leqq k\|ad(x)|_{\phi(A)}\|$$

for all $x\in B(H)$, where the left side is the distance of x and the commutant $\phi(A)'$ of $\phi(A)$ and $ad(x)(a)=xa-ax$.

The reader is referred to Christensen([4], [5]) for several results on property D_k.

THEOREM 9. Every k-similar C^*-algebra A has property $D_{ek/4}$. (in particular, D_k.)

PROOF. Let ϕ be a non-degenerate $*$-representation of A

on a Hilbert space H and let x be in $B(H)$. We consider for each positive real t the representation ϕ_t of A into $B(H\oplus H)$ given by

$$\phi_t(a)=\begin{pmatrix} \phi(a) & t\delta(a) \\ 0 & \phi(a) \end{pmatrix}, \qquad a\in A$$

where $\delta=ad(x)\circ\phi$. Using Lemma 3, we have

$$\begin{aligned}\|\phi_t(a)\| &= \left\|\begin{pmatrix} 1 & 0 \\ 0 & \phi(a) \end{pmatrix}\begin{pmatrix} 1 & t\delta(a) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \phi(a) & 0 \\ 0 & 1 \end{pmatrix}\right\| \\ &\leqq \left\|\begin{pmatrix} 1 & 0 \\ 0 & \phi(a) \end{pmatrix}\right\| \left\|\begin{pmatrix} 1 & t\delta(a) \\ 0 & 1 \end{pmatrix}\right\| \left\|\begin{pmatrix} \phi(a) & 0 \\ 0 & 1 \end{pmatrix}\right\| \\ &= \frac{1}{2}[\|t\delta(a)\|+\{\|t\delta(a)\|^2+4\}^{1/2}](\max\{1,\|\phi(a)\|\})^2 \\ &\leqq \frac{1}{2}\{t\|\delta\|+(t^2\|\delta\|^2+4)^{1/2}\}(\max\{1,\|a\|\})^3,\end{aligned}$$

hence

$$\|\delta_t\|\leqq\frac{1}{2}\{t\|\delta\|+(t^2\|\delta\|^2+4)^{1/2}\}.$$

By the k-similarity of A, ϕ_t it completely bounded and $\|\phi_t\|_{cb}\leqq\|\phi_t\|^k$. Then we have

$$\begin{aligned} t\|\delta\|_{cb} &\leqq \|\phi_t\|_{cb}\leqq\|\phi_t\|^k \\ &\leqq\left[\frac{1}{2}\{t\|\delta\|+(t^2\|\delta\|^2+4)^{1/2}\}\right]^k,\end{aligned}$$

so

$$\|\delta\|_{cb}\leqq\inf_{t\in R_+} t^{-1}\left[\frac{1}{2}\{t\|\delta\|+(t^2\|\delta\|^2+4)^{1/2}\}\right]^k.$$

The right side of above inequality attains a minimum value at $t=2/(k^2-1)^{1/2}\|\delta\|$. So $\|\delta\|_{cb}$ is dominated by the value $\frac{1}{2}(k+1(\{1+2/(k-1)\}^{(k-1)/2}\|\delta\|$. Then the function $\frac{1}{k}(k+1)\cdot\{1+2/(k-1)\}^{(k-1)/2}$ is monotone increasing for $k>1$, and it takes a limit value e at infinity. Therefore

$$||\delta||_{cb} \leqq \frac{e}{2}k||\delta||.$$

By [5, Proposition 2.1] we have

$$d(x, \phi(A)') = \frac{1}{2}||\delta||_{cb}$$
$$\leqq \frac{e}{4}k||\delta||(\leqq k||\delta||).$$

Thus the proof is compete.

REMARK. The above proof is based on [4, Theorem 3.2]. By this theorem we can get better estimation than [4], that is, any type II_1-factor with property Γ has property D_{30}.

Now we show that k-similarity implies property D_k. But we don't know whether property D_k imlies k-similarity or not. Though we shall show the following fact like proposition 6 for C^*-algebras with property D_k.

PROPOSITION 10. Let A and B be C^*-algebras.

(1) If $A \oplus B$ has property D_k, then so A and B have.

(2) If A and B have property D_k, then $A \oplus B$ has property D_{k+1}.

PROOF. (1) It is obvious.

(2) Let ϕ be a $*$-representation of $A \oplus B$ on a Hilbert space H and x be in $B(H)$. Then $p = \phi(1, 0)$ and $q = \phi(0, 1)$ are the orthogonal projections with $p + q = 1$. We denote by p and q, the projections of orthogonal subspaces H_1 and H_2 respectively. We define $*$-representations by $\phi_1 = \phi|_A$ on H_1 and $\phi_2 = \phi|_B$ on H_2 and we denote $\phi = \phi_1 \oplus \phi_2$.

For any $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in B(H_1 \oplus H_2)$ and $y = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} \in \phi(A \oplus B)' = \phi_1(A)' \oplus \phi_2(B)'$, we have

$$\left\|\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}-\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}\right\|=\left\|\begin{pmatrix} x_{11}-y_1 & 0 \\ 0 & x_{22}-y_2 \end{pmatrix}+\begin{pmatrix} 0 & x_{12} \\ x & 0 \end{pmatrix}\right\|$$
$$\leqq\left\|\begin{pmatrix} x_{11}-y_1 & 0 \\ 0 & x_{22}-y_2 \end{pmatrix}\right\|+\left\|\begin{pmatrix} 0 & -x_{12} \\ x_{21} & 0 \end{pmatrix}\right\|$$
$$=\max\{\|x_{11}-y_1\|,\ \|x_{22}-y_2\|\}+\left\|ad(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\|.$$

So we get

$$d(x,\phi(A\oplus B)')\leqq \max\{d(x_{11},\phi_1(A)'),\ d(x_{22},\phi_2(B)')\} + \|ad(x)|_{\phi(A\oplus B)}\|.$$

By the assumption of property D_k for A and B,

$$d(x,\phi(A\oplus B)')\leqq(k+1)\|ad(x)|_{\phi'(A\oplus B)}\|.$$

Thus we conclude $A\oplus B$ has property D_{k+1}.

ACKNOWLEDGEMENT. The authors would like to thank R. Ichihara for his helpful suggestion and Professor O. Takenouchi for his constant encouragement.

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Received March 10, 1988