

<研究報文>



## 單純한 메트릭스階乘 接近에 의한 高速아다마르變換

A Simple Matrix Factorization Approach to  
Fast Hadamard Transform

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### 要 約

高速아다마르變換은 Cooley-Tukey 알고리즘에 의해서 發表되어졌고, 그것은 메트릭스 分割 또는 階乘의 기술에 의한 것이다. 本 報文은 單純한 寄生메트릭스를 크로벡커 積에 의해 앞端과 연결시켜가면서 高速아다마르 變換을 보였다. 이것은 기존에 發表된 方法에 비해 쉽게 寄生메트릭스를 구할 수 있는 것을 確認했고 수학적으로 完全함을 증명했다.

### Abstract

The development of the FHT (fast Hadamard transform) was presented and based on the derivation by Cooley-Tukey algorithm. Alternately, it can be derived by matrix partitioning or matrix factorization techniques.

This paper proposes a simple sparse matrix technique by Kronecker product of successive lower Hadamard matrix. The following shows how the Kronecker product can be mathematically defined and efficiently implemented using a matrix factorization methods.

### 1. Introduction

The Hadamard transform has recently been applied in digital communication, the transmission of digital images, and also in pattern recognition for image processing and feature extraction. [1][5][7]

The elements of a Hadamard matrix take on values of plus and minus 1 only. This leads to simple implementation with electronic technology and simplifies the analysis by digital computer.

Furthermore, FHT can be used by factoring the Hadamard matrix. [1][2]. This in turn reduces

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the number of required operation and provides a faster computer implementation.

This paper introduces a basic Hadamard matrix partition and then successive Kronecker product [3][4]. The result of this method was easily shown to be the sparse matrix [4][6].

## 2. Sparse Matrix Representation

It is well known that the matrix factorization method has long been established.[3] Generally, to achieve 'In place' computation, the existing methods require a shuffle right after each operation has been performed.

In order to apply the FHT directly on the lower Hadamard matrix ( $[H]_2$ ) decomposition and still retain the property by Kronecker product use the following definition.

$$\text{def 1. } [H]_1 = [1], [H]_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix} \quad (1)$$

$$\text{def 2. } [I]_1 = [1], [I]_2 = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \quad (2)$$

The Hadamard matrix is represented by the sign for the matrix decomposition.

$$[H]_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix} = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \begin{bmatrix} + & + \\ + & - \end{bmatrix} = [I]_2 [H]_2 = [H]_2 [I]_2 \quad (3)$$

where + and - indicate +1 and -1, respectively.

$$\begin{aligned} [H]_4 &= \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} = \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\ &= ([H]_2 \otimes [I]_2) \times ([I]_2 \otimes [H]_2) = ([H]_2 [I]_2) \otimes ([I]_2 [H]_2) \end{aligned} \quad (4)$$

$$\text{or} \quad \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} = \begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{bmatrix} \begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{bmatrix} \quad (5)$$

$$\begin{aligned} [H]_8 &= ([H]_2 \otimes [I]_2) \otimes ([I]_2) \times ([I]_2 \otimes [H]_4) \\ &= ([H]_2 \otimes [I]_2^2) \times ([I]_2 \otimes [H]_2 \otimes [I])_2 \times ([I]_2^2 \otimes [H]_2) \\ &= \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \\ &\quad \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \\ &= \begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ + & 0 & - & 0 \\ 0 & + & 0 & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \times \begin{bmatrix} + & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \\ &\quad \times \begin{bmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \end{aligned} \quad (6)$$

and then

$$\begin{array}{c}
 \left[ \begin{array}{cccccc} + & + & + & + & + & + \\ + & - & + & - & + & - \\ + & + & - & - & + & + \\ + & - & - & + & + & - \\ + & + & + & - & - & - \\ + & - & + & - & + & - \\ + & + & - & - & + & + \\ + & - & - & + & - & + \end{array} \right] = \left[ \begin{array}{cccc} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & - \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \end{array} \right] \times \\
 \left[ \begin{array}{cccc} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{array} \right] \\
 \left[ \begin{array}{c} + 0 + 0 \\ 0 + 0 + 0 \\ + 0 - 0 \\ 0 + 0 - \\ + 0 + 0 \\ 0 + 0 + \\ 0 + 0 - \\ 0 + 0 - \end{array} \right] \times \left[ \begin{array}{c} + + 0 0 \\ + - 0 0 0 \\ 0 0 + + \\ 0 0 + - \\ + 0 + 0 \\ + - 0 0 \\ 0 0 0 + + \\ 0 0 + - \end{array} \right] \times \left[ \begin{array}{c} + + 0 0 \\ + - 0 0 \\ 0 0 0 + + \\ 0 0 + - \end{array} \right]
 \end{array} \tag{7}$$

$$[H]_{16} = ([H]_2 \otimes [I]_2^{[2]} \otimes [I]_2) \times ([I]_2 \otimes [H]_8) = ([H]_2 \otimes [I]_2^{[2]} \otimes [I]_2) \times ([I]_2 \otimes [H]_2 \otimes [I]_2^{[2]}) \\ \times ([I]_2^{[2]} \otimes [H]_2 \otimes [I]_2) \times ([I]_2^{[2]} \otimes [I]_2 \otimes [H]_2) \quad (8)$$

where  $[I]_2$  is the identity matrix of order  $2 \times 2$ , which leads directly to the fast algorithm.

and

$$[H]_{16} = \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[z]} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[z]} \\ \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[z]} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \times \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}^{[z]} \otimes \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \otimes \begin{bmatrix} + & + \\ + & - \end{bmatrix} \quad (9) \end{math>$$

$$\begin{array}{l}
\left[ \begin{array}{cccccc} + & 0 & 0 & 0 & + & 0 & 0 \\ 0 & + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & + & 0 & 0 & 0 & - \\ 0 & 0 & 0 & + & 0 & 0 & 0 \end{array} \right] \times \\
\left[ \begin{array}{cccccc} + & 0 & 0 & 0 & + & 0 & 0 \\ 0 & + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 & + \\ 0 & + & 0 & 0 & - & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & + & 0 & 0 & 0 & - \\ 0 & 0 & 0 & + & 0 & 0 & - \end{array} \right] \\
\left[ \begin{array}{cccccc} + & 0 & + & 0 & 0 & 0 & 0 \\ 0 & + & 0 & + & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & 0 & 0 \\ 0 & + & 0 & - & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + \\ + & 0 & - & 0 & 0 & 0 & - \\ 0 & + & 0 & - & 0 & 0 & - \end{array} \right] \times \\
\left[ \begin{array}{cccccc} + & 0 & + & 0 & 0 & 0 & 0 \\ 0 & + & 0 & + & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & 0 & 0 \\ 0 & + & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & 0 & 0 & + & 0 \end{array} \right] \\
\left[ \begin{array}{cccccc} + & + & 0 & 0 & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & + & 0 & 0 & 0 \\ 0 & 0 & + & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & 0 & + & - & 0 \\ 0 & 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & + & - \end{array} \right]
\end{array}$$

(10)

Then, letting  $[H]_N$  represent the sparse matrix order  $N(n=\log_2 N)$ , the recursive relationship is given by the expression:

$$\begin{aligned}[H]_N &= [H]_2 \otimes \underbrace{[I]_2 \otimes \dots \otimes [I]_2}_{\log_2 N - 1} \times [ ] \otimes [H]_{N/2} \\ &= ([H]_2 \otimes [I]_{N/2}) ([I]_2 \otimes [H]_{N/2})\end{aligned}\quad (11)$$

The proof of (11) is very simple. Using the algebra of Kronecker product [1][8] we have

$$[H]_N = [H]_2 \otimes [H]_{N/2} \quad (12)$$

$$\begin{aligned}[H]_N &= ([H]_2 \otimes [I]_{N/2}) ([I]_2 \otimes [H]_{N/2}) = ([H]_2 [I]_2) \otimes ([I]_{N/2} [H]_{N/2}) \\ &= [H]_2 \otimes [H]_{N/2}\end{aligned}\quad (13)$$

From (12), the right hand side of (13) is just  $[H]_N$  and the proof is complete.

The proposed method is based on matrix decomposition and ‘fork’ form in Fig. 1.

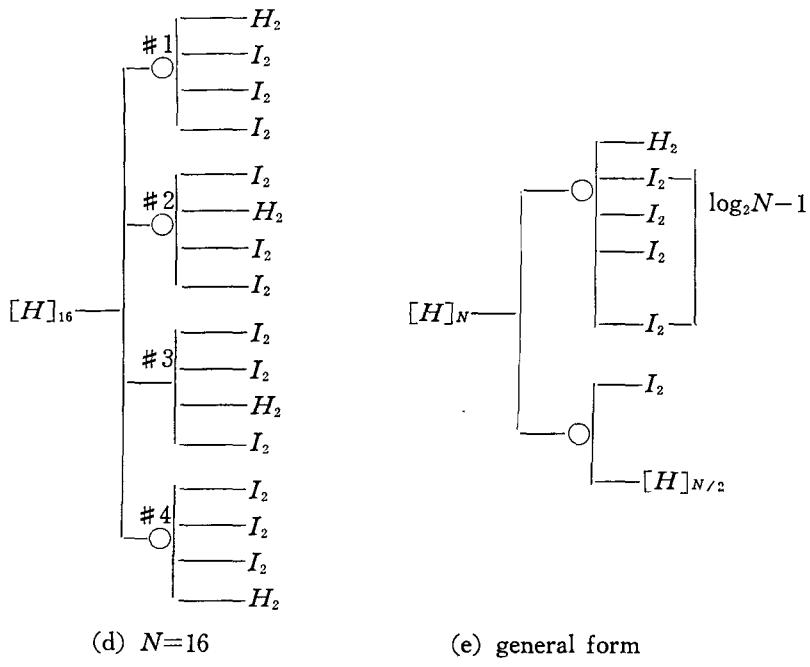
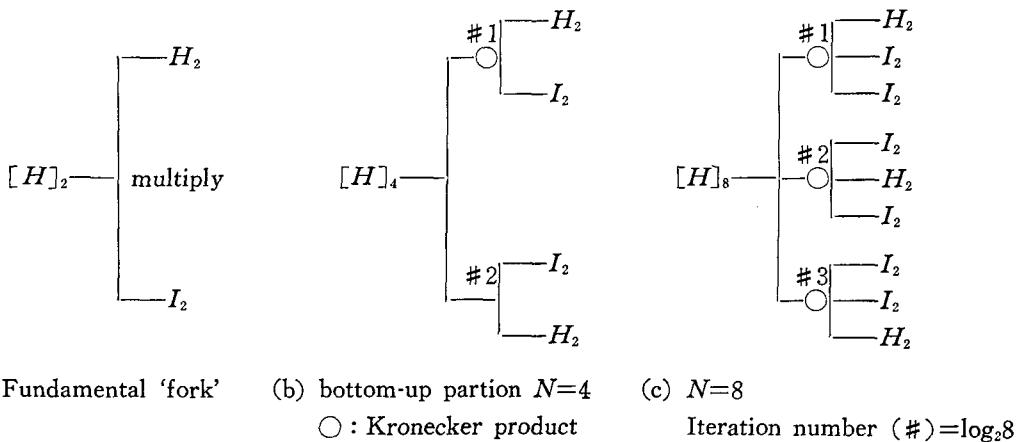


Fig. 1. Sparse matrix of FHT by Kronecker product

The procedure getting the sparse matrix of the FHT can now be summarized as follows;

Step 1 : Search the lower order of the Hadamard sparse matrix  $[H]_2$ ,  $[I]_2$  and iteration number (#).

Step 2 : Multiply the Kronecker product by the lower Hadamard Sparse matrix, according to the iteration number.

Step 3 : In general, multiply Kronecker product  $[H]_2$ ,  $[I]_{N/2}$  by  $[I]_2$ ,  $[H]_{N/2}$  according to iteration number.

The simple recursive relationship (11) can now be used to formulate a sparse-matrix decomposition of  $[H]_N$ . Expanding the second term in (11) with successively lower orders of the Hadamard matrix results in

$$[H]_N = \prod_{i=1}^k ([I] \otimes [H]_2 \otimes [I]_{N/2^i}). \quad (14)$$

Each matrix in the product form of (14) is sparse, in the sense that the  $i$ -th matrix has only two non-zero elements, +1 or -1, in each row and column. Thus the transform operation requires  $kN$  operations. For illustration  $[H]_8$  is depicted below according to the decomposition in (14).

$$[H]_8 = \left( \begin{array}{ccccccccc} + & + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - & - \\ + & + & - & - & + & + & - & - & - \\ + & - & - & + & + & - & - & + & + \\ + & + & + & + & - & - & - & - & - \\ + & - & + & - & - & + & - & + & + \\ + & + & - & - & - & - & + & + & + \\ + & - & - & + & - & + & + & - & - \end{array} \right) = \left( \begin{array}{ccccccccc} + & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & - \end{array} \right) \\ \left( \begin{array}{ccccc} + & 0 & + & 0 & \\ 0 & + & 0 & + & 0 \\ + & 0 & - & 0 & \\ 0 & + & 0 & - & \\ & & + & 0 & + & 0 \\ & 0 & 0 & + & 0 & + \\ & & + & 0 & - & 0 \\ & 0 & 0 & + & 0 & - \end{array} \right) \left( \begin{array}{ccccc} + & + & 0 & 0 & \\ + & - & 0 & 0 & 0 \\ 0 & 0 & + & + & \\ 0 & 0 & + & - & \\ & & + & + & 0 & 0 \\ & 0 & & + & - & 0 & 0 \\ & & 0 & 0 & + & + & 0 \\ & 0 & 0 & + & - & - & 0 \end{array} \right)$$

#### 4. Conclusion

An algorithm for the FHT, using Kronecker product, is developed on matrix decomposition.

Most of the papers published to date, use a sparse matrix of basic  $[H]_4$  decomposition, but we showed here lower order of Hadamard  $[H]_2$  and then have represented them in general form.

Compared with other methods, this seems to be simple, basic, clear, and straightforward.

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## 案　　内

本會에서는 87 年 會員名單 發刊을 為하여 會員 여러분들의 정확한 人的事項을 파악코자 합니다.

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