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A Continuous Review $(S\text{-}1, S)$ Inventory Policy in which Depletion is due to Demand and Loss of Units

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ABSTRACT

A stochastic model for an inventory system in which depletion of stock takes place due to random demand as well as random loss of items is studied under the assumption that the intervals between successive unit demands, as well as those between successive unit losses, are independently and identically distributed random variables having negative exponential distribution with respective parameters. We have derived the steady state probability distribution of the net inventory level assuming negative exponential delivery time under the continuous review $(S\text{-}1, S)$ inventory policy. Also we have derived the total expected cost expression and necessary conditions to be satisfied for an optimal solution.

1. INTRODUCTION

In certain real world situations it is appropriate to order units one at a time as demanded. This can be true, for example, if the demand for the item is very low or the item is very expensive.

This policy is called a continuous review $(S\text{-}1,S)$ inventory (i.e., one-for-one-ordering) policy and fills demands on a first-come first-served basis. It means that a reorder is placed whenever a demand occurs and the inventory position (i.e., the amount on hand plus on order minus backorders) remains constant.

Innumerable papers have been written analyzing mathematical models for describing $(S\text{-}1, S)$ inventory policies [Das (1971), Feeney (1966), Gross (1971), Hadley (1963), Sherbrooke (1968)]. Invariably, it was assumed implicitly that once units enter into inventory, they last forever or else they expire after only a single planning period. Though this assumption is not altogether unreasonable or unjustifiable in cases where planning over a short run is essentially unaffected by obsolescence, there is a wide variety

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of situations that arise in food industry, drug industry and in the area of health administration in which the perishable nature of the inventory has to be taken into account in developing optimal ordering policies.

Some researchers have considered this aspect of the inventory problem [Nahmias (1975), Nahmias (1976), Nahmias (1978)] but have assumed that the items, like foodstuffs, photographic films, drugs and pharmaceuticals and blood, have fixed life time. However there exists a significant class of problems for which the life time of the items in the inventory is not deterministic and the assumption of random life time of the items would be more appropriate (e.g., the electronic industry).

Recently, authors [Choi and Kim (1985), Kumaraswamy and Sankarasubramanian (1981)] have considered a problem in which the depletion of inventory level is due to random demand as well as random failure (or loss) of items proportional to on hand inventory. But, in their papers, they assumed instantaneous replenishment of order under a continuous review (s, S) inventory policy.

In this paper, we consider a $(S\text{-}1, S)$ problem under the assumption that replenishment time is negative exponentially distributed with parameter, that demands occur in a Poisson manner with parameter and that life time distribution of each item in inventory is negative exponential with parameter. We have derived the steady state probability distribution of the net inventory level and the total cost expression.

2. ANALYSIS OF THE PROBLEM

Assumptions

(1) The demands occur in a Poisson manner with parameter μ.
(2) The life time distribution of each item in inventory is negative exponential with parameter λ .
(3) The replenishment time distribution is negative exponential with parameter r.

Notations

μ : average demand rate
λ : average failure (loss) rate
r : average item-replenishment rate
S : stock level specified
K : ordering cost per order
C : variable procurement cost per unit
h : holding cost per unit per unit time
π : backorder cost per unit backordered per unit time.

Let $H(t)$ be the net inventory level at an arbitrary time t. Let $P(n, t)=P\{H(t)=n\}$ denote the probability that there are exactly n units in stock at time t.

Changes in the net inventory level are caused by demands, losses of units in inventory, and arrivals of on-order inventory. In a time dt, the net inventory moves from state n+1 to state n if a demand or a loss occurs. In a time dt, the net inventory moves from state n to n+1 if an on-order inventory arrives.

Thus we have the diagram representing the transitions of the net inventory level shown in Figure 1. The following equations represent the transitions shown in Figure 1.

For $n=S$ and $t \geqq 0$,

$$P(S,\ t+dt)=[1-(\mu+S\lambda)dt]\,P(S,\ t)+r\,dt\,P(S-1,\ t).$$

For $\quad 0 \leqq n \leqq S-1 \quad$ and $\quad t \geqq 0,$

$$P(n,\ t+dt)=[1-(\mu+n\lambda)dt-(S-n)r\,dt]\,P(n,\ t)$$

$$+[(S-n+1)r\,dt]\,P(n-1,t)+[\mu+(n+1)\lambda]\,dt\,P(n+1,\ t).$$

For $\quad n<0 \quad$ and $\quad t \geqq 0,$

$$P(n,\ t+dt)=[1-\mu dt-(S-n)\,r\,dt]\,P(n,\ t)$$

$$+[(S-n+1)\,r\,dt]\,P(n-1,\ t)+\mu\,dt\,P(n+1,\ t).$$

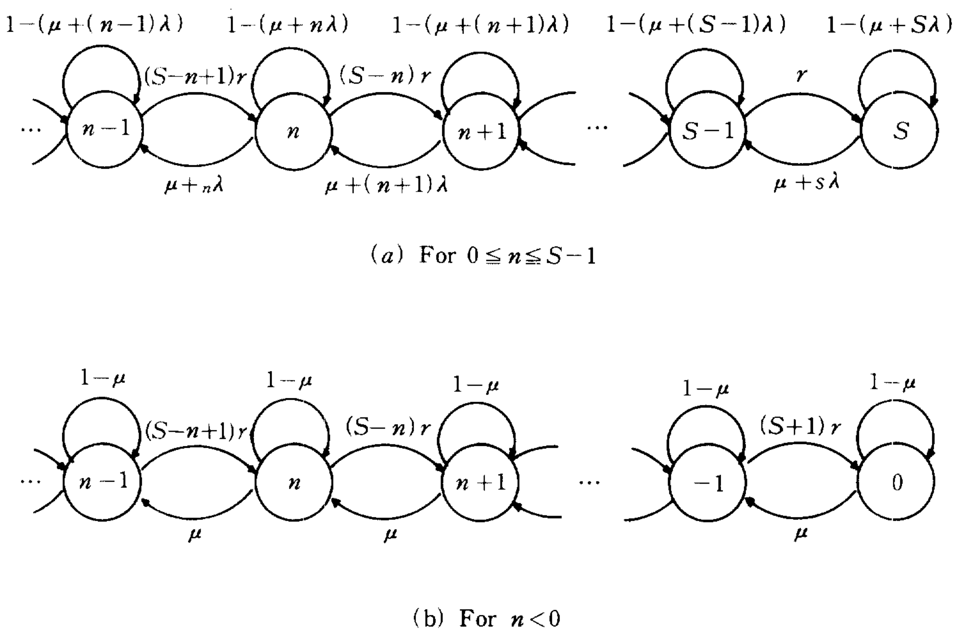

Figure 1. Transition diagram for the net inventory level.

Transposing, dividing by dt, and letting $dt \to 0$, these equations can be written as

$$\frac{dP(S,\ t)}{dt}=-(\mu+S\lambda)\,P(S,\ t)+rP(S-1,\ t),\quad n=S. \tag{1}$$

$$\frac{dP(n,t)}{dt} = -[(\mu+n\lambda)+(S-n)r]\,P(n,t)+(S-n+1)r\,P(n-1,t)$$

$$+[\mu+(n+1)\lambda]\,P(n+1,t), \qquad 0\leqq n\leqq S-1 \tag{2}$$

$$\frac{dP(n,t)}{dt} = -[\mu+(S-n)r]\,P(n,t)+(S-n+1)r\,P(n-1,t)$$

$$+\mu P(n+1,t), \qquad n<0. \tag{3}$$

Let P_n be the probability that in the steady state there are exactly n net inventory, that is

$$P_n = \lim_{t\to\infty} P(n,t) = \lim_{t\to\infty} P\{H(t)=n\}.$$

The following steady state difference equations are obtained from (1), (2), and (3).

$$-(\mu+S\lambda)P_s + r\,P_{s-1}=0, \qquad n=S \tag{4}$$

$$-[(\mu+n\lambda)+(S-n)r]\,P_n+(S-n+1)\,rP_{n-1}$$

$$+[\mu+(n+1)\lambda]\,P_{n+1}=0, \qquad 0\leqq n\leqq S-1. \tag{5}$$

$$-[\mu+(S-n)r]\,P_n+(S-n+1)\,rP_{n-1}$$

$$+\mu\;P_{n+1}=0, \qquad n<0. \tag{6}$$

Solving recursively the system of Eqs. (4) and (5), we have

$$P_{s-1} = \frac{\mu+S\lambda}{r}\,P_s,$$

$$P_{s-2} = \frac{\mu+(S-1)\lambda}{2r}\,P_{s-1} = \frac{\mu+(S-1)\lambda}{2r}\;\frac{\mu+S\lambda}{r}\,P_s.$$

In general, we find that

$$P_n = \frac{\prod_{i=1}^{s-n}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\,r^{s-n}}\,P_s, \qquad 0\leqq n\leqq S-1. \tag{7}$$

Solving recursively the system of Eqs. (5) and (6), we have

$$P_{-1} = \frac{\mu}{(S+1)r} P_0,$$

$$P_{-2} = \frac{\mu}{(S+2)r} P_{-1} = \frac{\mu^2}{(S+2)(S+1)r^2} P_0.$$

In general, we find that

$$P_n = \frac{\mu^{-n}}{(S-n)(S-n-1)\ldots(S+1)r^{-n}} P_0$$

$$= \frac{\mu^{-n}\prod_{i=1}^{s}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\, r^{s-n}} P_s, \qquad n<0. \tag{8}$$

The value of P_s is determined by recorgnizing that

$$\sum_{n=-\infty}^{s} P_n = 1. \tag{9}$$

Thus, substituting Eqs. (7) and (8) into Eq. (9),

$$P_s = \{\mu - S\prod_{i=1}^{s}[\mu+(S-(i-1))\lambda]e^{\mu/r} - \sum_{n=0}^{s}\frac{\mu^{-n}\prod_{i=1}^{s}[\mu+(S-(i-1))\lambda]}{(S-n)!\, r^{s-n}}$$

$$+\sum_{n=0}^{s}\frac{\prod_{i=1}^{s-n}[\mu+(S-(i-1))\lambda]}{(S-n)!\, r^{s-n}}\}^{-1} \tag{10}$$

Thus, the expected on hand inventory in the steady state is

$$E(H) = \sum_{n=1}^{s} nP_n = \sum_{n=1}^{s} n\frac{\prod_{i=1}^{s-n}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\, r^{s-n}} P_s \tag{11}$$

The expected backorder quantity in the steady state is

$$E(B) = \sum_{n=-\infty}^{-1}(-n)P_n = \sum_{n=-\infty}^{-1}(-n)\frac{\mu^{-n}\prod_{i=1}^{s}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\, r^{s-n}} P_s \tag{12}$$

Let V_n be the time taken for transition from state n to state n-1. Then, the following Lemma can be proved.

[Lemma] The probability density function of $f_{V_n}(t)$ for $n \geq 0$ is given by

$$f_{V_n}(t) = (\mu + n\lambda)\exp[(\mu + n\lambda)t], \quad t \geq 0.$$

(Proof) Let Y be the time until a demand occurs. And let Z_i be the time until loss (or failure) of the ith unit among the n units of item on hand. Then V_n takes on the value equal to the minimum of Y, $Z_1, Z_2, \ldots, Z_n$, that is,

$V_n = \min\{Y, Z_1, Z_2 \ldots, Z_n\}$. Now note that, for any $t \geq 0$,

$$P\{V_n > t\} = P\{Y > t, Z_1 > t, Z_2 > t, \ldots, Z_n > t\}$$

$$= P\{Y > t\} P\{Z_1 > t\} \ldots P\{Z_n > t\}$$

$$= \exp(-\mu t)\exp(-\lambda t) \ldots \exp(-\lambda t)$$

$$= \exp\{-(\mu + n\lambda)t\}.$$

Thus, V_n has an exponential distribution with parameter ;

$\mu + n\lambda$. (Q, E. D.)

Also, it is obvious that for $n < 0$ (in case when losses can not occur), V_n has an exponential distribution with parameter μ.

Let $E(V)$ denote the expected elapsed time between two successive orders. Then, $E(V)$ is obtained by averaging over the net inventory, i.e.,

$$E(V) = \sum_{n=-\infty}^{s} P_n E(V_n) = \sum_{n=-\infty}^{-1} \{ \frac{\mu^{-n} \prod_{i=1}^{s} [\mu + (S - (i-1))\lambda]}{(S-n)!\, r^{s-n}} P_s \frac{1}{\mu} \}$$

$$+ \sum_{n=0}^{s} \{ \frac{\prod_{i=1}^{s-n} [\mu + (S-(i-1))\lambda]}{(S-n)!\, r^{s-n}} P_s \frac{1}{\mu + n\lambda} \}$$

$$= \{ \mu^{-s-1} \prod_{i=1}^{s} [\mu + (S-(i-1))\lambda] e^{\mu/r} - \sum_{n=0}^{s} \frac{\mu^{-n-1} \prod_{i=1}^{s} [\mu + (S-(i-1))\lambda]}{(S-n)!\, r^{s-n}}$$

$$+ \sum_{n=0}^{s} \frac{\prod_{i=1}^{s-n} [\mu + (S-(i-1))\lambda]}{(S-n)!\, r^{s-n}} \frac{1}{\mu + n\lambda} \} P_s. \qquad (13)$$

Let the procurement cost consists of fixed cost of K per order and a variable cost C per unit. Under steady state conditions, the expected procurement cost per unit time is $(K+C)\,E(V)$. The expected holding cost per unit time is $hE(H)$, and the expected backorder cost per unit time is $\pi E(B)$. From Eqs. (11), (12) and (13), the total expected cost per unit time is given by

$$C(S)=\frac{K+C}{E(V)}+hE(H)+\pi E(B)$$

$$=(K+C)\{\mu^{-s-1}\prod_{i=1}^{s}[\mu+(S-(i-1))\lambda]e^{\mu/r}-\sum_{n=0}^{s}\frac{\mu^{-n-1}\prod_{i=1}^{s}[\mu+(S-(i-1))\lambda]}{(S-n)!\,r^{s-n}}$$

$$+\sum_{n=0}^{s}\frac{\prod_{i=1}^{s-n}[\mu+(S-(i-1))\lambda]}{(S-n)!\,r^{s-n}}\ \frac{1}{\mu+n\lambda}\}^{-1}\ P_s^{-1}+h\sum_{n=1}^{s}n\frac{\prod_{i=1}^{s-n}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\,r^{s-n}}\ P_s$$

$$+\pi\sum_{n=-\infty}^{1}(-n)\ \frac{\mu^{-n}\prod_{i=1}^{s}\{\mu+[S-(i-1)]\lambda\}}{(S-n)!\,r^{s-n}}\ P_s \tag{14}$$

The above expression is to be minimized with respect to S.
Observing that S is a positive integer, the optimum value S satisfies the following conditions;

$$C(S^*)-C(S^*-1)\leqq 0, \tag{15}$$

$$C(S^*)-C(S^*+1)\leqq 0. \tag{16}$$

3. NUMERICAL EXAMPLE

As an illustration to the above developed model, consider a system with the following parameter values; K = \$5 per order, C = \$100 per unit, h = \$20 per unit per unit time, π = \$2200 per unit per unit time, $\mu = 10$, $\lambda = 2$, and $r = 15$. Applying the inequalities (15) and (16), we find that the optimal value of S is $S^* = 2$. From Eq. (14), $C(S^*=2)$ =\$1425.33. Table 1 shows the optimal value S with different values of λ. The results of sensitivity analysis show that the effect of loss rate is substantial on the total cost and optimal value of S.

Table 1. Optimal value of S varying with λ .

λ	S^*	$c(S^*)$
0	3	\$ 1108.97
1	2	\$ 1303.00
2	2	\$ 1425.33
3	2	\$ 1525.25
4	2	\$ 1605.22
5	2	\$ 1668.21
6	2	\$ 1717.00
7	2	\$ 1754.16
8	1	\$ 1777.05
9	1	\$ 1794.36

4. CONCLUDING REMARKS

The earlier works on the perishable problems assumed that the life of product is fixed and known with certainty.

However, in many circumstances, this assumption is not appropriate. In this paper, we analyzed the problem, in which, life time of items is random under the continuous review (S-1, S) inventory policy.

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