Kyungpook Math. J.
Volume 27, Number 2
December, 1987

LOWER RADICALS OF Γ-RINGS

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Abstract: In this note we introduce the concept of a lower radical for Γ-rings. As an application we also characterise the prime radical introduced by Barnes [1] as a lower radical. Furthermore it is shown that the prime radical can also be determined by the class of all semiprime Γ-rings.

1. Preliminaries

In this note we consider only the Γ-rings in the sense of Barnes [1]. A class $\boldsymbol{R}$ of Γ-rings (fixed Γ) is called a radical class if $\boldsymbol{R}$ satisfies:

(A) $\boldsymbol{R}$ is homomorphically closed.

(B) If every nonzero homomorphic image of a Γ-ring has at least one nonzero $\boldsymbol{R}$-ideal, then the Γ-ring itself belongs to $\boldsymbol{R}$.

The unique maximal $\boldsymbol{R}$-ideal of a Γ-ring M, denoted by $\boldsymbol{R}(M)$, is called the $\boldsymbol{R}$-radical of M and $\boldsymbol{R}(M/\boldsymbol{R}(M))=0$. A Γ-ring M is called $\boldsymbol{R}$-semisimple if $\boldsymbol{R}(M)=0$. $\boldsymbol{SR}$ will denote the class of all $\boldsymbol{R}$-semisimple Γ-rings.

Furthermore we say that a class $\boldsymbol{A}$ of Γ-rings is hereditary if $M\in\boldsymbol{A}$ and I an ideal of M (denoted by $I\lhd M$), implies that $I\in\boldsymbol{A}$. If $\boldsymbol{A}$ is a hereditary class of Γ-rings, then

$$U\boldsymbol{A}=\{\Gamma\text{-rings } M \mid M/I\notin\boldsymbol{A} \text{ for every } M\neq I\lhd M\}$$

is a radical class, called the upper radical class determined by $\boldsymbol{A}$.

Let M be a Γ-ring. $Q\lhd M$ is called a prime ideal of M if for $A,B\lhd M$ and $A\Gamma B\subseteq Q$, implies $A\subseteq Q$ or $B\subseteq Q$. Q is called a semiprime ideal of M if for $U\lhd M$ and $U\Gamma U\subseteq Q$, implies $U\subseteq Q$. M is called a prime (semiprime) Γ-ring if 0 is a prime (semiprime) ideal of M.

An element a of a Γ-ring M is nil if $a(\Gamma a)^n=0$, $n\in N$. M is nil if every element of M is nil. $S\subseteq M$ is called nilpotent if $S(\Gamma S)^n=0$, $n\in N$. S is called zero if $S\Gamma S=0$.

A Γ-ring M is called a simple zero ring if $M\Gamma M=0$ and M and 0 are the only two ideals of M. M is called a simple prime ring if $M\Gamma M=M$ and M and 0 are the only two ideals of M.

AMS (MOS) classification: 16A78, 16A48 and 16A21

A class $\boldsymbol{A}$ of Γ-rings is called a special (weakly special) class if $\boldsymbol{A}$ satisfies

(a) $\boldsymbol{A}$ consists of prime (semiprime) Γ-rings.

(b) $\boldsymbol{A}$ is hereditary.

(c) $\boldsymbol{A}$ is essentially closed i.e. for a Γ-ring M, I an essential ideal in M and $I\in\boldsymbol{A}$, implies that $M\in\boldsymbol{A}$.

If $\boldsymbol{R}=\boldsymbol{U}\boldsymbol{A}$ where $\boldsymbol{A}$ is a special (weakly special) class, then $\boldsymbol{R}$ is called a special (weakly special) radical. A radical class is called supernilpotent if it is hereditary and contains all the nilpotent Γ-rings.

2. The lower radical class for Γ-rings

DEFINITION 2.1. Let $\boldsymbol{B}$ be a nonempty class of Γ-rings such that $\boldsymbol{B}$ is homomorphically closed. Define $\boldsymbol{B}=\boldsymbol{B}_1$. Assuming that $\boldsymbol{B}_t$ has been defined for every ordinal $1\leq t<v$, we define $\boldsymbol{B}_v$ to be the class of all Γ-rings M such that every nonzero homomorphic image of M contains a nonzero ideal $I\in\boldsymbol{B}_t$ for some $t<v$.

The proofs of the following two lemmas are minor modifications of the proofs of the corresponding theorems in ordinary ring theory, and will be omitted.

LEMMA 2.2. *If $\boldsymbol{LB}=\bigcup_t \boldsymbol{B}_t$, then $\boldsymbol{LB}$ is a radical class, and it is the smallest radical class containing $\boldsymbol{B}$.*

LEMMA 2.3. *If $\boldsymbol{B}$ is a hereditary, homomorphically closed class of rings, then $\boldsymbol{LB}$ is hereditary.*

$\boldsymbol{LB}$ is called the lower radical class determined by $\boldsymbol{B}$. Let $Q\triangleleft I\triangleleft M$ where M is a Γ-ring. In [2] it was shown that $\overline{Q}\Gamma\overline{Q}\Gamma\overline{Q}\subseteq Q$ where $\overline{Q}=Q+Q\Gamma M+M\Gamma Q+M\Gamma Q\Gamma M$, is the ideal of M generated by Q. This result can be considered as Andrunakievic's lemma for Γ-rings. Before we give a concrete application, we need the following two lemmas.

LEMMA 2.4. *Let $\boldsymbol{A}$ be a hereditary, homomorphically closed class of Γ-rings which contains all the zero rings. Then $\boldsymbol{LA}=\boldsymbol{A}_2$.*

PROOF. Let $M\in\boldsymbol{A}_3$. Every nonzero homomorphic image M' of M has a nonzero ideal $I\in\boldsymbol{A}_2$. Since $I\in\boldsymbol{A}_2$, every nonzero homomorphic image of I (hence

I itself) has a nonzero ideal $B \in \boldsymbol{A}_1 = \boldsymbol{A}$. Let $\overline{B}$ be the ideal of M' generated by B.

Furthermore $B \lhd \overline{B} \lhd M'$. From Andrunakievic's lemma we have $\overline{B}\Gamma\overline{B}\Gamma\overline{B} \subseteq B \in \boldsymbol{A}$. Since $\boldsymbol{A}$ is hereditary, we have $\overline{B}\Gamma\overline{B}\Gamma\overline{B} \in \boldsymbol{A}$. If $\overline{B}\Gamma\overline{B} = 0$, then $\overline{B} \in \boldsymbol{A}$ since $\boldsymbol{A}$ contains all the zero rings. If $\overline{B}\Gamma\overline{B} \neq 0$, but $\overline{B}\Gamma\overline{B}\Gamma\overline{B} = 0$, then $\overline{B}\Gamma\overline{B}\Gamma\overline{B}\Gamma\overline{B} = 0$, so that $\overline{B}\Gamma\overline{B}$ is a nonzero zero ring and hence $\overline{B}\Gamma\overline{B} \in \boldsymbol{A}$. If $\overline{B}\Gamma\overline{B}\Gamma\overline{B} \neq 0$, then M' contains a nonzero ideal in $\boldsymbol{A}$. So, in any case, M' contains a nonzero ideal in $\boldsymbol{A}$. Therefore $M \in \boldsymbol{A}_2$ and hence $\boldsymbol{A}_3 = \boldsymbol{A}_2$. So $\boldsymbol{L}\boldsymbol{A} = \boldsymbol{A}_2$.

If $\boldsymbol{K}$ is the class of all nilpotent Γ-rings, it is easy to verify that $\boldsymbol{K}$ satisfies the conditions of the previous lemma. Therefore, we have.

LEMMA 2.5. *Let $\boldsymbol{K}$ be the class of all the nilpotent Γ-rings. Then $\boldsymbol{L}\boldsymbol{K} = \boldsymbol{K}_2$.*

3. Application to the prime radical of Γ-rings

In order to give an application of the lower radical construction of Γ-rings, we will show that $\boldsymbol{L} = \{$All nilpotent Γ-rings$\} = \boldsymbol{U}\boldsymbol{P} = \boldsymbol{U}\boldsymbol{C}$, where $\boldsymbol{P} = \{$All prime Γ-rings$\}$ and $\boldsymbol{C} = \{$All semiprime Γ-rings$\}$.

First, however, we need the following two lemmas.

LEMMA 3.1. *Let S be a semiprime ideal of the Γ-ring M.*

(i) $A\Gamma B \subseteq S$ if and only if $B\Gamma A \subseteq S$, for any ideals A and B of M.

(ii) $A(\Gamma B)^n \subseteq S$ if and only if $A\Gamma B \subseteq S$, for any ideals A and B of M and $n \in IN$.

PROOF. (i) $A\Gamma B \subseteq S$ implies $B\Gamma A\Gamma B\Gamma A \subseteq B\Gamma S\Gamma A \subseteq S\Gamma A \subseteq S$. Since S is a semiprime ideal, we have $B\Gamma A \subseteq S$.

The converse can be proved in a similar way.

(ii) If $A\Gamma B \subseteq S$, it follows that $A(\Gamma B)^n \subseteq S$ for any $n \in IN$.

Conversely, let $A(\Gamma B)^n \subseteq S$. If $n = 1$, then $A\Gamma B \subseteq S$. So, suppose $n > 1$, then

$\Longrightarrow \{[B\Gamma]^{n-1} A\Gamma B\}\Gamma\{[B\Gamma]^{n-1} A\Gamma B\} \subseteq S$

$\Longrightarrow [B\Gamma]^{n-1} A\Gamma B \subseteq S$ since S is semiprime

$\Longrightarrow \{[B\Gamma]^{n-1} A\}\Gamma\{[B\Gamma]^{n-1} A\} \subseteq S$

$\Longrightarrow [B\Gamma]^{n-1} A \subseteq S$; S is semiprime

i.e. $B\Gamma B\Gamma \ldots B\Gamma A \subseteq S$

i.e. $(B\Gamma B\Gamma \ldots . B)\Gamma A \subseteq S$

From (i) we have $A\Gamma(B\Gamma B\Gamma \ldots B) \subseteq S$

$\Longrightarrow A(\Gamma B)^{n-1} \subseteq S$

The proof is completed by induction.

LEMMA 3.2. *A Γ-ring M is semiprime if and only if M has no nonzero nilpotent ideals.*

PROOF. Let M be semiprime. Suppose $0 \neq I \lhd M$ such that $I(\Gamma I)^n = 0$, $n \in IN$. Since (0) is a semiprime ideal of M, we have from (ii) of the previous lemma that $I\Gamma I = 0$ and again, since (0) is a semiprime ideal of M, we have $I = 0$; a contradiction. Therefore M has no nonzero nilpotent ideals.

Conversely, suppose M has no nonzero nilpotent ideals. If $A\Gamma A = 0$ for some $A \lhd M$, then $A = 0$. This implies that M is semiprime.

Now if $\boldsymbol{K} = \{$All the nilpotent Γ-rings$\}$, we have from lemma 2.5 that $\beta = \boldsymbol{LK} = \boldsymbol{K}_2 = \{M$ a Γ-ring$|$Every $0 \neq M/I$ has a nonzero nilpotent ideal$\}$.

Let M be any Γ-ring and denote by $\beta(M)$ the sum of all the β-ideals of M.

THEOREM 3.3 *If $\boldsymbol{C}$ is the class of all the semiprime Γ-rings, then $\boldsymbol{S}\beta = \boldsymbol{C}$.*

PROOF. Let $M \in \boldsymbol{S}\beta$. Then $\beta(M) = 0$. Therefore M has no nonzero β-ideals and hence no nonzero nilpotent ideals. From lemma 3.2 we have that $M \in \boldsymbol{C}$.

Conversely, let $M \in \boldsymbol{C}$. Suppose $M \notin \boldsymbol{S}\beta$, then $\beta(M) \neq 0$ and hence $\beta(M)$ has a nonzero nilpotent ideal N. Let $\overline{N}$ be the ideal of M generated by N. From Andrunakievic's lemma for Γ-rings, we have $\overline{N}\Gamma\overline{N}\Gamma\overline{N} \subseteq N$. If $0 = \overline{N}\Gamma\overline{N}\Gamma\overline{N} = \overline{N}(\Gamma\overline{N})^2$, then $\overline{N}$ is a nonzero nilpotent ideal of M; a contradiction with $M \in \boldsymbol{C}$. If $\overline{N}\Gamma\overline{N}\Gamma\overline{N} \neq 0$ then $\overline{N}\Gamma\overline{N}\Gamma\overline{N}$ is a nonzero nilpotent ideal of M since $\overline{N}\Gamma\overline{N}\Gamma\overline{N} \subseteq N$, and N is nilpotent. Again this is a contradiction since $M \in \boldsymbol{C}$. Therefore $M \in \boldsymbol{S}\beta$ and the theorem proved.

As in the case of ordinary rings, we also have in the case of Γ-rings that every radical class $\boldsymbol{R}$ is the upper radical class determined by the semisimple class $\boldsymbol{SR}$, i.e. $\boldsymbol{R} = \boldsymbol{USR}$.

From theorem 3.3 we now have

THEOREM 3.4. *$\beta = \boldsymbol{UC}$, where $\boldsymbol{C}$ is the class of all semiprime Γ-rings.*

The following lemma is easy to verify.

LEMMA 3.5. *M/I is a prime (semiprime) Γ-ring if and only if I is a prime (semiprime) ideal in the Γ-ring M.*

we are now in a position to prove that β is also the upper radical class determined by all the prime Γ-rings M.

THEOREM 3.6. *$\beta=\boldsymbol{UP}$, where $\boldsymbol{P}$ is the class of all prime Γ-rings.*

PROOF. Since $\boldsymbol{P}\subseteq\boldsymbol{C}$, we have that $\boldsymbol{UC}=\beta\subseteq\boldsymbol{UP}$. Suppose $\beta\neq\boldsymbol{UP}$. Then there exists $0\neq M\in\boldsymbol{UP}$, $M\not\in\beta$, i.e. $\beta(M)\neq M$. Therefore there exists $x\in M$, $x\not\in\beta(M)$. Let X be the ideal of M generated by x. Suppose $X(\Gamma X)^n\not\subseteq\beta(M)$ for every $n\in N$. Let $U=\{J\triangleleft M\,|\,X(\Gamma X)^k\not\subseteq J\supseteq\beta(M)$, for all $k=1,2,\cdots\}$. There exists, according to Zorn's lemma, an ideal W of M, maximal to the property $X(\Gamma X)^k\not\subseteq W$. We want to show that W is a prime ideal of M. Let L_1 and L_2 be two ideals of M such that $L_1\Gamma L_2\subseteq W$. If $W\subset L_1+W$ and $W\subset L_2+W$, then there exist $r,t\in\boldsymbol{N}$ such that $X(\Gamma X)^r\subseteq L_1+W$ and $X(\Gamma X)^t\subseteq L_2+W$. Hence $[X(\Gamma X)^r]\Gamma[X(\Gamma X)^t]\subseteq(L_1+W)\Gamma(L_2+W)$ i.e. $X(\Gamma X)^P\subseteq L_1\Gamma L_2+L_1\Gamma W+W\Gamma L_2+W\Gamma W\subseteq W$; a contradiction. Hence $L_1+W=W$ or $L_2+W=W$. This implies that $L_1\subseteq W$ or $L_2\subseteq W$, i.e. W is a prime Γ-ideal of M. From lemma 3.5 we have that $M/W\in\boldsymbol{P}$; a contradiction with $M\in\boldsymbol{UP}$. Hence, there exists $k\in\boldsymbol{N}$ such that $X(\Gamma X)^k\subseteq\beta(M)$. Now we have that $0\neq[\beta(M)+X]/\beta(M)\triangleleft M/\beta(M)\in\boldsymbol{S}\beta$. But $\{[\beta(M)+X]/\beta(M)\}\ \{\Gamma[\beta(M)+X]/\beta(M)\}^k=0$. Therefore $M/\beta(M)$ has a nonzero nilpotent ideal; a contradiction with $M/\beta(M)\in\boldsymbol{S}\beta$. So, $M\in\beta$ and $\beta=\boldsymbol{UP}$.

In [2], theorem 2.2, it was proved that $\boldsymbol{P}$ is a special class and hence that $\beta=\boldsymbol{UP}$ is a special radical. It is easy to check that if $\boldsymbol{K}$ is the class of all nilpotent Γ-rings, then $\boldsymbol{K}$ is hereditary and homomorphically closed. From lemma 2.3 it follows that $\boldsymbol{LK}=\beta$ is also hereditary. Therefore $\beta=\boldsymbol{UP}=\boldsymbol{UC}$ is a supernilpotent radical class.

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