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# A CONDITION FOR THE COMMUTATIVITY OF RINGS

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**Abstract**: In the present paper a result [10] of the authors has been generalized as follows: Let $l, m, n$ be fixed positive integers and $R$ be a semi prime ring in which $[(xy)^l, (xy)^m-(yx)^n]=0$ for all $x, y\in R$, then $R$ is commutative.

## 1. Introduction

Recently the authors [10] improved a theorem of Abu-Khuzam and Yaqub [1] which in turn generalizes a well-known result due to Israel N. Herstein [5] that a division ring $D$ in which for all $x, y$ in $D$, $xy-yx$ is central must be commutative. Infact we proved: If $R$ is a semi simple ring in which for each $x, y$ in $R$ there exists a positive integer $n=n(x, y)$ for which either $[z, (xy)^n+(yx)^n]=0$ or $[z, (xy)^n-(yx)^n]=0$, for all $z\in R$, where $[x, y]=xy-yx$, then $R$ is commutative. Our present objective is generalize the above result as follows:

THEOREM. *Let $l, m, n$ be fixed positive integers and $R$ be a semi prime ring in which* $[(xy)^l, (xy)^m-(yx)^n]=0$, for all $x, y\in R$, *then $R$ is commutative.*

This theorem is an extension of the result of Belluce, Herstein and Jain [4] and also that of Hazar Abu-Khuzam [2].

## 2. In preparation for the proof of our theorem we first establish the following lemmas:

LEMMA 2.1. *Let $l, m, n$ be fixed positive integers and $R$ be a semi prime ring in which* $[(xy)^l, (xy)^m-(yx)^n]=0$. *for all $x, y\in R$, then $R$ has no non-zero nilpotent element.*

PROOF. Let $a\in R$ such that $a^2=0$. By using the hypothesis of the lemma we get

$$(ax)^l(xa)^n=[(ax)^l, (ax)^m-(xa)^n]=0, \text{ for all } x\in R$$

Hence

$$\begin{aligned}(ax)^{l+n+1}&=\{(ax)^{l+n}a+(ax)^l(xa)^n\}x\\&=\{a(xa+x)\}^l\{(xa+x)a\}^n x=0\end{aligned}$$

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If $aR\neq 0$, then $aR$ is a nonzero nilright ideal satisfying the identity $z^{l+n+1}=0$, for all $z$ in $aR$. By lemma 1.1 of [6], $R$ has a nonzero nilpotent ideal. This is a contradiction since $R$ is a semi prime. Thus $aR=0$ i.e. $aRa=0$, which forces that $a=0$.

Further lemma 1.1.1 of [8] combining with the above result atonce yields the following.

LEMMA 2.2 *Let $l, m, n$ be fixed positive integers and $R$ be a prime ring in which $[(xy)^l, (xy)^m-(yx)^n]=0$, for all $x, y \in R$, then $R$ has no nonzero zero-devisors.*

LEMMA 3.2. *Let $l, m, n$ be fixed positive integers and $R$ be a division ring in which $[(xy)^l, (xy)^m-(yx)^n]=0$, for all $x, y \in R$, then $R$ is commutative.*

PROOF. Using the hypothesis, we get easily

$$[(xy)^l, (yx)^n]=0, \text{ for all } x, y \in R.$$

Now let $c$ be a nonzero element of $R$ and $S$ be a division subring of $R$ generated by the set of elements $\{a^{-1}ca \mid a(\neq 0)\in R\}$. Since $S=x^{-1}Sx$ for all nonzero $x\in R$, $S$ is invarient to all inner automorphisms and we have by Cartan-Brauer-Hua theorem [9, p. 186] either $S=R$ or $S\subseteq Z(R)$, the centre of the ring $R$. But as shown just above for any nonzero $a\in R$ there exist positive integers $l$ and $n$ such that

$$[(a.a^{-1}c)^l, (a^{-1}c.a)^n]=0$$
$$\text{i.e. } [c^l, a^{-1}c^n a]=0$$

Now if $S\subseteq Z(R)$ then obviously $c^n\in Z(R)$, and $R$ is commutative by Herstein's Theorem [7]. On the other hand if $S=Z(R)$, then $c\in Z(R)$ and again $R$ is commutative. Thus in every case $R$ is commutative.

PROOF OF THEOREM. Since $R$ is semi prime ring in which $[(xy)^l, (xy)^m-(yx)^n]=0$ for all $x, y \in R$ and for some positive integers $l, m, n$, $R$ is isomorphic to a subdirectsum of prime rings $R_\alpha$ each of which as a homomorphic image of $R$ satisfies the hypothesis placed on $R$. Therefore we can assume that $R$ is prime. Thus by lemma 2.2, $R$ contains no zero devisors. Hence by strengthening of Posner's theorem [Il, Cordlary 1], $R$ can be embedded in a division ring satisfying the same polynomial identity and by lemma 2.3, $R$ is commutative.

The following example shows that the condition that ring $R$ to be semi prime is not superfluous.

EXAMPLE. Let $R$ be the subring generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in the ring of $3\times 3$ matrices over $Z_2$, the ring of integer modulo 2. For all integers $l>1$, $m>1$ and $n>1$, $[(xy)^l, (xy)^m-(yx)^n]=\dot{0}$ holds in $R$. However, $R$ is not commutative.

REMARKS. The Baer's example [3] of nil semi prime ring establishes that the above result can not be extended to the case when $l, m$ and $n$ are arbitrary depending on the pair of elements $x, y$.

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