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ON THE (m, K, n)-IDEALS OF ASSOCIATIVE RINGS

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Let A be an associative ring. Let A^0 cenote the operator 1 even in case, if A does not have an unity element $e\in A$. Z denotes the ring of rational integers, and $B\cdot C$ denotes the additive subgroup generated by all products $b\cdot c$, where $b\in B$ and $c\in C$. A subring S of A is called an (m, K, n)-ideal, if the inclusion

$$S^m\cdot A\cdot S^K\cdot A\cdot S^n\subseteq S$$

is valid, for arbitrary non-negative rational integers m, K, n. It must be remarked that (m, K, n)-ideals of semigroups were introduced and discussed by S. Lajos [6].

Following S. Lajor-F.A. Szász [8], a subring S of a ring A is called an (m, n)-ideal of A, if $S^m\cdot A\cdot S^n\subseteq S$ holds. Obviously, every (m, n)-ideal of A is also an (m, K, n)-ideal, by $A\cdot S^K\cdot A\subseteq A$, and thus by $S^m\cdot A\cdot S^K\cdot A\cdot S^n\subseteq S^m\cdot A\cdot S^n\subseteq S$ we have shown our assertion. But the converse, in general is not true. The converse statement holds if $A^2=A$ is valid, since then the $(m, 0, n)$-ideal coincides with the (m, n)-ideal, being true

$$S^m\cdot A\cdot S^0\cdot A\cdot S^n=S^m\cdot A^2\cdot S^n=S^m\cdot A\cdot S^n\subseteq S$$

A Particular case of $A^2=A$ is, if A is von Neumann-regular, strongly regular, weakly regular, or biregular. For these definitions see N. Jacobson [1], or F.A. Szász [11].

THEOREM 1. *A subring S of an associative ring is an (m, K, n)-ideal of A if and only if S is an (m, n)-ideal of a $(0, K, 0)$-ideal of A.*

PROOF. First we assume that S is a $(0, K, 0)$-ideal of the ring A, and T is an (m, n)-ideal of S. Then one has

$$A\cdot S^K\cdot A+S^2\subseteq S$$

and $T^mST^n+T^2\subseteq T\subseteq S$.

Now these inclusions imply evidently

$$T^m\cdot A\cdot T^K\cdot A\cdot T^n\subseteq T^mAS^k\cdot A\cdot T^n\subseteq T^mST^m\subseteq T,$$

i.e. T is an (m, K, n)-ideal of A, indeed.

Conversely, let S be an (m, K, n)-ideal of A. Then we shall show that S is an (m, n)-ideal of the $(0, K, 0)$-ideal, generated by S, of A. Obviously

$$(*) \qquad \{S\}_{(0,K,0)} = S + A \cdot S^K \cdot A,$$

where the left side of our equation (*) is the $(0, K, 0)$-ideal, generated by S, of A. Thus we have

$$S^m \cdot \{S\}_{(0,k,0)} \cdot S^n \subseteq S^m A S^k A \cdot S^n + S^{m+n-1} \subseteq S,$$

and thus Theorem 1 is proved.

DEFINITION 2. (see S. Lajos [ ]). A subring S of a ring A is said to be an "interior ideal", if it is a $(0, 1, 0)$-ideal that is $A \cdot S \cdot A \in S$ holds.

COROLLARY 3. *A subring S of a ring A is a $(0, 1, 1)$-ideal of A if and only if S is a left ideal of an interior ideal of A.*

COROLLARY 4. *A subring S of a ring A is an $(1, 1, 1)$-ideal of A if and only if S is a bi-ideal of an interior ideal.*

REMARK 5.1. Bi-ideal S of A means an $(1, 1)$-ideal of A. For a detailed discussion of bi-ideals of rings, see S. Lajos-F. A.. Szász [7].

REMARK 5.2. Evidently, every two-sided ideal is an interior ideal, but the converse, in general, is not true, as the following example shows:

EXAMPLE 6. Let F be an arbitrary field, and $A = F_6$ the full ring of matrics of type 6×6 over the field F. Its matrix unities let be denoted by $E_{i,j}$ $(i, j = 1, 2, 3, 4, 5$ and $6)$, i.e. we have

$$E_{i,j} \cdot E_{k,l} = \delta_{j,k} \cdot E_{i,l}$$

where $\delta_{j,K}$ is the Kronecker delta. If we put

$$a_1 = E_{1,2}\,;\; a_2 = E_{1,3} + E_{t,1}\,;\; a_3 = E_{1,4}\,;\; a_4 = E_{1,5} + E_{4,1} + E_{6,3}$$
$$\text{and } a_5 = E_{1,6} + E_{4,1} + E_{5,2},$$

then the additive subgroups

$$Z \cdot a_3\,;\; Za_5\,;\; Za_2 + Za_3\,;\; Za_1 + Za_4 \text{ and } Za_1 + Za_5$$

all are subrings, even interior ideals, but not twosided ideals of $A = F_6$.

THEOREM 7. *Let A be an associative ring, S an (m, n)-ideal of A. Then any $(0, K, 0)$-ideal T of A is an $(m, m+k+n, n)$-ideal of A.*

PROOF. Obviously

$$S^m A S^n + S^2 \subseteq S$$

and $S \cdot T^K \cdot S + T^2 \subseteq T \subseteq S$.

Now we shall show that the subring T of A is an $(m, m+K+n, n)$-ideal of A. Namely from the above inclusions we have

$$T^m\cdot A\cdot T^{m+K+n}\cdot A\cdot T^n=(T^mAT^n)\cdot T^K\cdot(T^mAT^n)$$
$$\subseteq(S^mAS^n)\cdot T^K(S^mAS^n)\subseteq AT^kA\subseteq T.$$

Therefore T is an $(m,\ m+k+n,\ n)$-ideal of A, indeed.

COROLLARY 8. *Assume that A is a ring, T is a bi-ideal of A, and S is an interior ideal of T. Then S is a $(1,3,1)$-ideal of A.*

COROLLARY 9. *Assume that A is a ring, L is a left ideal of A, and S is an interior ideal of L. Then S is a $(0,2,1)$-ideal of A.*

THEOREM 10. *Let A be a ring, S an (m,k,n)-ideal of A. They any (p,q,r)-ideal of S is an $(m+p+q+1,\ K,\ n+r)$-ideal resp. an $(m+p,\ K,\ n+q+r+1)$-ideal of A.*

PROOF. Evidently hold:

$$S^m\cdot A\cdot S^K\cdot A\cdot S^n+S^2\subseteq S$$

and

$$T^p\cdot S\cdot T^q\cdot S\cdot T^r+T^2\subseteq T\subseteq S.$$

Therefore

$$T^{m+p+q+1}\cdot A\cdot T^k\cdot AT^{n+r}=T^{p+q+1}(T^m\cdot A\cdot T^k\cdot AT^n)T^r$$
$$\subseteq T^{p+q+1}\cdot S\cdot T^r\subseteq T^p\cdot ST^qST^r\subseteq T.$$

COROLLARY 11. *Assume that A is a ring, S is an (m,K,n)-ideal of A. Then any interior ideal T of S is an $(m,k,n+2)$-ideal, resp. $(m+2,\ K,\ n)$-ideal of A.*

COROLLARY 12. *Assume that A is a ring, B is an interior ideal of A. Then any (m,k,n)-ideal* T *of S is an $(m+k+1,\ 1,\ n)$-ideal resp. $(m,1,\ n+k+1)$-ideal of A.*

Recently S Lajos-G. Szász [8] *have introduced the notion of (p,q,r)-regularity of semigroups. In a similar way, we say that a ring A is (p,q,r)-regular, if there exist elements x and y of A for every $a\in A$ such that*

$$a=a^p\cdot x\cdot a^q\cdot y\cdot a^r$$

holds. Moreover, following S. Lajos [6], *we say that an (m,k,n)-ideal of A is "complete" if $S^m\cdot A\cdot S^K\cdot A\cdot S^n=S$ holds.*

THEOREM 13. *A ring A is (p,q,r)-regular if and only if every (p,q,r)-ideal of A is complete.*

PROOF. Let A be a (p,q,r)-regular ring, S a (p,q,r)-ideal of A and $s\in S$.

Then $s=s^p\times s^q\cdot y\cdot s^r\in S^p\cdot A\cdot S^q\cdot A\cdot S^r$ implies $S\subseteq S^p\cdot A\cdot S^q\cdot A\cdot S^r\subseteq S$, whence it follows that the (p,q,r)-ideal is complete.

Conversely, if every (p,q,r)-ideal S of A is complete then for every $s\in S$ we have

$$s\in\{s\}_{(p,q,r)}=T=T^p\cdot A\cdot T^q\cdot A\cdot T^r\subseteq s^p\cdot A\cdot s^q\cdot A\cdot s^r,$$

which means the (p,q,r)-regularity of the ring A.

COROLLARY 14. *A ring A is von Neumann-regular if and only if every* $(1,1,1)$*-ideal S of A is complete.*

COROLLARY 15. *A ring A is intraregular if and only if every* $(0,2,0)$*-ideal of A is complete.*

We recall that a ring A is strongly regular if for every element $a\in A$ the inclusion $a\in a^2\cdot A$ holds.

The characterization of strongly regular rings with the help of 29 equivalent conditions can be found e.g. in part II of F. A. Szász [10].

Now, the following result holds:

COROLLARY 16. *The following eleven conditions for a ring are pairwise equivalent*:

(1) *A is strongly regular.*

(2) *Every* $(1,0,2)$*-ideal of A is complete.*

(3) *Every* $(2,0,1)$*-ideal of A is complete.*

(4) *Every* $(2,0,2)$*-ideal of A is complete.*

(5) *Every* $(1,1,2)$*-ideal of A is complete.*

(6) *Every* $(2,1,1)$*-ideal of A is complete.*

(7) *Every* $(2,1,2)$*-ideal of A is complete.*

(8) *Every* $(1,2,2)$*-ideal of A is complete.*

(9) *Every* $(2,2,1)$*-ideal of A is complete.*

(10) *Every* $(2,2,2)$*-ideal of A is complete.*

(11) *For every $m,n\in Z$ such that $m+n\geqq 3$ and $m,n\geqq 1$, every (m,n)-ideal of A is complete.*

PROBLEM 17. Investigate the generalized (m,K,n)-ideals S of the ring A such that $S^mAS^KAS^n\subseteq S$ holds, but S is only a subgroup of A^+, but, in general, it is not a subring of A!

PROBLEM 18. Let C be a class of rings A. Investigate the almost (m, K, n)-Amitsur-Kurosh radical classes C, satisfying.

18. 1) C is homomorphically closed

18. 2) If every nonzero homomorphic image A' of the ring A contains a C-subring S, which is an (m, K, n)-ideal S of A, then $A \in C$ holds.

PROBLEM 19. Investigate those (m, K, n)-strongly Amitsur-Kurosh radical classes $C(m, K, n)$ such that in every ring A, the corresponding radical $C(m, K, n)(A)$ of A contains every subring, which is $C(m, K, n)$-radical ring!

PROBLEM 20. The same problem, as Problems 18 and 19, but for generalized (m, K, n)-ideals of the ring A, instead of ordinary (m, K, n)-ideals of A.

PROBLEM 21. The same, as problem 19, but for supernilpotent, special and subidempotent radical classes.

REMARK 22. For these concepts see e.g. F. A. Sfász [11].

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