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ON THE ABSOLUTE CONVERGENCE OF LACUNARY VECTOR VALUED FOURIER COEFFICIENTS SERIES

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Abstract: In this article the absolute convergence of lacunary Fourier Coefficients Series is studied for Hilbert space valued functions. The considered functions are assumed to be of either the modulus of continuity or the modulus of smoothness of order l which are considered only at a fixed point in $[-\pi,\pi]$. On the other hand for values in weakly sequentially complete Banach space, the lacunary Fourier coefficients series is strongly unconditionally convergent. The results obtained here are a kind of a generalization of theresults due to Kandil [4].

1. Introduction

Let $x(t)$ denotes in general-a strongly continuous periodic vector valued function of real variable t of period 2π with values in a Banach space X with the norm $\|.\|$.

The lacunary Fourier series of $x(t)$ is defined as follows:

$$x(t)=\sum_{k=-\infty}^{\infty} C_{n_k} e^{in_k t} \tag{1.1}$$

If $x(t)$ is integrable in the sence of Pettis [2], we can write the lacunary Fourier Coefficients as:

$$C_{n_k}=\frac{1}{2\pi}\int_{-\pi}^{\pi} x(t)e^{-in_k t}dt,\ C_{n_0}=0,\ K=\pm 1,\ \pm 2,\cdots \tag{1.2}$$

Let us introduce the definitions and notations which will be used in the sequel.

(I) On anology to the scaler valued case, we define here the modulus of continuity $\omega(x,\ \eta)$ of $x(t)$ by:

$$\omega(x,\eta)=\sup_{|h|<\eta} \|x(t_0+h)-x(t_0-h)\| \tag{1.3}$$

(II) The modulus of smoothness $\omega_l(x,\ \eta)$ of $x(t)$ of order l $(l\in N)$ at the point $t_0\in[-\pi,\pi]$ are defined by:

$$\omega_l(x,\ \eta)=\sup\|\sum_{j=1}^{l}(-1)^{l-j}\binom{l}{j}x(t_0+(2j-l)h)\| \tag{1.4}$$

(III) The series $(1-1)$ satisfies the lacunary condtion

$$(n_{k+1}-n_k) > CF(n_k),\ C>0 \tag{1.5}$$

where $F(n_k)$ increases to ∞ as $k\to\infty$,

$$F(n_k) < n_k \text{ for all } k\in N.$$

With $F(n_k)=n_k$, this condition gives the famous Hadamard lacunary condition

$$\lim_{k\to\infty} \inf \frac{C_{n_{k+1}}}{C_{n_k}} > 1 \tag{1.6}$$

Throughout the paper $\delta=8\pi/(CF(n_N))$, where N is a natural number, and

$$D=\{t: |t-t_0|<\delta\}$$

Let H be a Hilbert space H and let $B_2(H,\ D)$ denote the space of H-valued Pettis integrable functions x defined on D [3].

To formulate our results, we introduce, the following lemmas which will be used in what follows:

LEMMA 1.1. *The modulus of continuity $\omega(x,\eta)$ defined as (1.3) is such that:*
(i) For each positive λ, we have

$$\omega(x,\ \lambda\eta)\leq(\lambda+1)\ \omega(x,\ \eta)$$

(ii) $w(x,\ 0)=0$

(iii) $\omega(x,\ \eta)$ is increasing.

PROOF. Obvious.

LEMMA 1.2. [7]. *If $u_n\geq 0$ $(n\in N)$, $u_n\neq 0$, and $F(u_n)$ is a function such that $F(0)=0$, $F(u)$ increasing and concave. Then*

$$\sum_{n=1}^{\infty} F(u_n) < 2\sum_{n=1}^{\infty} F\left(\frac{u_n+u_{n_{+1}}+\cdots}{n}\right)$$

LEMMA 1.3. *Let.*

(i) $x(t)\in B_2(H,\ D)$ for some D,

(ii) C_{n_k} be a lacunary Fourier coefficients

$$K=0,\pm1,\pm2,\cdots$$

(iii) $(n_{_{k+1}}-n_k)\geq 8\pi\delta^{-1}$ *for all* k $\qquad$ (1.7)

Then:

$$(i)\ \sum_{k=-\infty}^{\infty}\|C_{n_k}\|^2\leq 8\delta^{-1}\int_D\|x(t)\|^2dt \tag{1.8}$$

$$(ii)\ \sum_{|n_k|\geq n_N}\|C_{n_k}\|^2\leq C\left(\omega\left(x,\ \frac{A}{F(n_{_N})}\right)\right)^2, \tag{1.9}$$

or more generally

$$\sum_{|n_k|\geq n_N} \|C_{n_k}\|^2 \leq C(\omega_l(x, \frac{B}{F(n_N)})^2 \tag{1.10}$$

where $C>0$ and l is an odd natural number.

PROOF. The proof for the scaler valued case as given in [6].
Works just well in the vector valued situation.

2. Main results

THEOREM 2.1. *If*:

(1) *$x(t)$ is strongly continuous periodic vector valued function with values in a Hilbert space H defined on some D.*

(2) $(n_{k+1}-n_k)>CF(n_k)$, $C>0$

$$\text{(3)} \sum_{k=1}^{\infty} \frac{(\omega(x, \frac{A}{F(n_k)}))^\beta}{k^{\beta/2}} < +\infty, \quad 0<\beta\leq 1 \tag{2.1}$$

where $\omega\left(x, \frac{A}{F(n_k)}\right)$ is as in (1.3) *with η replaced by $A|F(n_k)$, $A=24\pi/C+\pi$.*

Then:

$$\sum_{k=-\infty}^{\infty} \|C_{n_k}\|^\beta < +\infty. \tag{2.2}$$

PROOF. Since $x(t)$ is strongly continuous on D, then we have:

$$x(t+h)-x(t-h) = \sum_{k=-\infty}^{\infty} C_{n_k}(e^{in_k} - e^{-in_kh})e^{in_kt}.$$

Applying parseral's identity [3], one gets:

$$\int_D \|x(t+h)-x(t-h)\|^2 dt = \sum_{k=-\infty}^{\infty} |e^{in_kh} - e^{-in_kh}|^2 \|C_{n_k}\|^2$$
$$= 4 \sum_{k=-\infty}^{\infty} \|C_{n_k}\|^2 \sin^2 n_k h$$

Hence from (1.8), we obtain:

$$4 \sum_{k=-\infty}^{\infty} \|C_{n_k}\|^2 \sin^2 n_k | h \leq 8\delta^{-1} \int_D \|x(t+h)-x(t-h)\|^2 dt \tag{2.3}$$

Integrating both sides of (2.3) with respect to h over $\left(0, \frac{\pi}{n_N}\right)$, one gets:

$$4 \sum_{k=-\infty}^{\infty} \|C_{n_k}\|^2 \int_0^{\pi/n_N} \sin^2 |n_k| h dh \leq \frac{CF(n_N)}{\pi} \int_0^{\pi/n_N}$$
$$dh\left(\int_{t_0-8/CF(n_N)}^{t_0+8/CF(n_N)} \|x(t+h)-x(t-h)\|^2 dt \right). \tag{2.4}$$

We see that if $|n_k|\geq n_N$, then:

$$\int_0^{\pi/n_N} \sin^2|n_k|h\,dh=\frac{1}{|n_N|}\int_0^{(|n_k|/n_N)\pi}\sin^2 t\,dt$$

$$>\frac{1}{n_N([|n_k|/n_N]+1)}\int_0^{[|n_k|/n_N]\pi}\sin^2 t\,dt$$

$$=\frac{1}{n_N}\,\frac{[|n_k|/n_N]}{1+[n_k|/n_N]}\cdot\frac{\pi}{2}$$

$$\geq\frac{1}{n_N}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4n_N}, \tag{2.5}$$

Using $1\leq[|n_k|/n_N]\leq|n_k|/n_N\leq 1+[|n_k|/n_N]$, where [ ] denotes the integral part.

Also by (1.7), using $\pi/n_N\leq\pi/F(n_N)$, and observing from (2.4) that $t\in[t_0-\delta,\ t_0+\delta]$, we obtain:

$$\begin{aligned}\|x(t+h)-x(t-h)\|&=\|x(t_0-\delta+\lambda+h)-x(t_0)+x(t_0)-x(t_0-\delta+\lambda-h)\|\\&\leq 2\,\omega(x,\ \delta+\lambda+h),\qquad 0<\lambda<2\,\delta\\&\leq 2\,\omega(x,\ 3\,\delta+h)\\&\leq 2\,\omega\Big(x,\ \frac{24\pi}{CF(n_N)}+\frac{\pi}{n_N}\Big)\\&=2\pi\Big(x,\ \frac{A}{F(n_N)}\Big),\ A=\pi+24\pi/C\end{aligned} \tag{2.6}$$

Using (2.6) and (2.5), we get from (2.4) that:

$$\sum_{|n_k|\geq n_N}\|C_{n_k}\|^2\leq C\Big(\omega\Big(x,\ \frac{A}{F(n_N)}\Big)\Big)^2, \tag{2.7}$$

So, we have:

$$\big(\sum_{|n_k|\geq n_N}\|C_{n_k}\|^2\big)\leq C\Big(\omega\Big(x,\ \frac{A}{F(n_N)}\Big)\Big)^\beta,\qquad 0<\beta\leq 1 \tag{2.8}$$

Applying Lemma (1.2) with $u_k=\|C_{nk}\|^2$, $k\in Z$ and $F(u)=u^{\beta/2}$, and using (2.8), we have:

$$\sum_{k=-\infty}^{\infty}\|C_{n_k}\|^\beta\leq 2\sum_{k=1}^{\infty}F(\|C_{n_k}\|^2)\leq 4\,C\sum_{k=1}^{\infty}\frac{\Big[\omega\Big(x,\ \frac{A}{F(n_k)}\Big)\Big]^\beta}{k^{\beta/2}}<+\infty$$

This shows that:

$$\sum_{k=-\infty}^{\infty}\|C_{n_k}\|^\beta<+\infty,$$

and this completes the proof of Theorem 2.1.

THEOREM 2.2. *If*:

(1) *$x(t)$ satisfies the all conditions of Theorem* 2.1.

(2) $\sum_{k=1}^{\infty} \frac{\left(\omega_l\left(x, \frac{B}{F(n_k)}\right)\right)^\beta}{k^{\beta/2}} < \infty$

where $w_l\left(x, \frac{B}{F(n_k)}\right)$ *is as in* (1.4) *with* η *replaced by* $\frac{B}{F(n_k)}$ *in which* $B = 8\pi/C + \pi$ *and* l *is an odd number.*

Then:

$$\sum_{k=-\infty}^{\infty} \|C_{n_k}\|^\beta < +\infty, \qquad 0 < \beta \leq 1$$

PROOF. Applying the inequality (1.10) instead of (1.9) in the proof of theorem 2.1 and proceeding analogously, this theorem is proved.

COROLLARY. *If*:

(1) *$x(t)$ is strongly continuous vector valued function on D with values in H.*

(2) $\{n_k\}$ *satisfies the Hadamard condition* (1.6)

(3) $\sum_{k=1}^{\infty} \frac{\left(\omega_l\left(x, \frac{B}{n_k}\right)\right)^\beta}{k^{\beta/2}} < +\infty, \qquad 0 < \beta \leq 1$

Then:

$$\sum_{k=-\infty}^{\infty} \|C_{n_k}\|^\beta < \infty$$

REMARK. With $l=2$, $\beta=1$, $B=1$, without the lacunary condition and with modulus of continuity $\omega_2\left(x, \frac{1}{K}\right)$ on the whole interval $[-\pi, \pi]$ instead of at the point to, the corollary is equivalent to the results due to Kandil [4].

The following theorem gives the behaviour of a lacunary Fourier Coefficients series when values of a vector valued function $x(t)$ in a weakly sequentially complete Banach space.

THEOREM 2.3. *If*:

(1) *$x(t)$ is a periodic vector valued function on $[-\pi, \pi]$ with period 2π and with values in a weakly complete Banach space.*

(2) *$X(t)$ is strongly continuous, and satisfies*

$$\sum_{k=1}^{\infty} \frac{\left(\omega\left(x, \frac{A}{F(n_k)}\right)\right)^\beta}{k^{\beta/2}} < +\infty \qquad 0 < \beta \leq 1$$

(3) $(n_{k+1}-n_k)>CF(n_k)$, $\quad C>0$

Then the series $\sum_{k=-\infty}^{\infty} C_{n_k}$ *is strongly unconditionally convergent.*

PROOF. Let X^* be the dual space of a Banach space X. Consider the scaler function $x^*x(t)$, where x^* is a linear functional in X^* with $\|x^*\|=1$.

The lacunary Fourier Coefficients series of $x^*x(t)$ are $x^*C_{n_k}$, $(k=0, \pm 1, \pm 2, \cdots)$. Applying Parseval's identity to $x^*x(t)$, we get:

$$4\sum_{k=-\infty}^{\infty}|x^*C_{n_k}|^2\sin^2|n_k|h=\int_D|x^*(x(t+h)-x(t-h))|^2dt$$

$$<\int_D\|x(t+h)-x(t-h)\|^2dt$$

Repeating the steps of the proof of Theorem 2.1 one gets.

$$\sum_{k=-\infty}^{\infty}|x^*C_{n_k}|^\beta<\infty, \qquad 0<\beta\leq 1$$

So, any subsequence of the partial sums of the series $\sum_{k=-\infty}^{\infty} C_{n_k}$ is weakly convergent. Using the given condition the space values is weakly sequentially complete Banach space, we deduce that the series $\sum_{k=-\infty}^{\infty} C_{n_k}$ is weakly unconditional convergent

Applying the fact that in a Banach space each weakly unconditional convergent series is strongly unconditional convergent [2]. Hence the theorem is proved.

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