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A NOTE ON ANALYTIC P-VALENT FUNCTIONS

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Abstract. Let $A(p)$ be the class of functions

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

which are analytic in the unit disk U. Further let $D^{n+p-1}f(z)$ denote the Hadamard product of $z^{p}/(1-z)^{n+p}$ and $f(z)$. In this paper, we shall define the class $S_{n,p}(\alpha,\ \beta)$ of functions $f(z)\in A(p)$ satisfying the condition

$$\mathrm{Re}\left\{\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n}{n+p}\right)^{\alpha}\left(\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-\frac{n+1}{n+p+1}\right)^{\beta}\right\}>0$$

for $p\in N$, $n>-p$, $z\in U$ and α and β are real numbers. The object of the present paper is to show a property of $S_{n,p}(\alpha,\ \beta)$.

Let $A(p)$ denote the class of functions

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N=\{1,2,3,\cdots\})$$

which are analytic in the unit disk $U=\{z:\ |z|<1\}$. Let $f*g(z)$ denote the Hadamard product of two functions

$$f(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

and

$$g(z)=z^{p}+\sum_{n=1}^{\infty}b_{p+n}z^{p+n} \qquad (p\in N),$$

that is,

$$f*g(z)=z^{p}+\sum_{n=1}^{\infty}a_{p+n}b_{p+n}z^{p+n}.$$

Further let

$$D^{n+p-1}f(z)=\frac{z^{p}}{(1-z)^{n+p}}*f(z) \qquad (z\in U) \tag{1}$$

for $p\in N$ and $n>-p$.

With this symbol $D^{n+p-1}f(z)$, N. S. Sohi [5] defined the classes T_{n+p-1} of

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functions $f(z)\in A(p)$ which satisfy the condition

$$\text{(2)} \qquad \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}\right\}>\frac{n}{n+p} \qquad (z\in U)$$

for $p\in N$ and $n>-p$.

In particular, T_0 is the class of functions $f(z)\in A(p)$ which satisfy the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$$

and we can see that such functions are p-valent by T. Umezawa [6].

In this paper, we shall introduce the following classes $S_{n,p}(\alpha,\ \beta)$ by using the symbol $D^{n+p-1}f(z)$.

DEFINITION. Let

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the class $A(p)$ and

$$\text{(3)} \qquad P_{n,p}(f(z)\ ;\ \alpha,\ \beta)=\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n}{n+p}\right)^{\alpha}$$

$$\mathrm{X}\left(\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-\frac{n+1}{n+p+1}\right)^{\beta} \qquad (z\in U),$$

where $p\in N$, $n>-p$ and α and β are real numbers. We say that $f(z)$ belongs to the class $S_{n,p}(\alpha,\ \beta)$ if $f(z)$ satisfies the condition

$$\text{(4)} \qquad \mathrm{Re}\{P_{n,p}(f(z)\ ;\ \alpha,\ \beta)\}>0 \qquad (z\in U).$$

The powers appearing in (3) are meant as principal values. We can observe that $S_{n,p}(1,\ 0)=T_{n+p-1}$ and $S_{n,p}(0,\ 1)=T_{n+p}$.

Recently, R.M. Geol and N. S. Sohi [1], [2] and S. Owa [3], [4] studied other classes of p-valent functions.

We now state and prove our theorem for the class $S_{n,p}(\alpha,\ \beta)$.

THEOREM. *Let $p\in N$, $n>-p$ and $0\leqq t\leqq 1$. Then we have*

$$\text{(5)} \qquad S_{n,p}(\alpha,\ \beta)\cap T_{n+p-1}\subset S_{n,p}((\alpha-1)t+1,\ \beta t).$$

PROOF. Let the function

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the class $S_{n,p}(\alpha,\ \beta)\cap T_{n+p-1}$ and

$$\text{(6)} \qquad \left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n}{n+p}\right)^{\alpha}\left(\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-\frac{n+1}{n+p+1}\right)^{\beta}=V_{n,p}(z).$$

Then, by means of $f(z)\in S_{n,p}(\alpha,\ \beta)$, $\mathrm{Re}\{V_{n,p}(z)\}>0$ for $z\in U$.
Further let

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n}{n+p}=U_{n,p}(z). \tag{7}$$

Then, by using $f(z)\in T_{n+p-1}$, we have $\mathrm{Re}\{U_{n,p}(z)\}>0$ for $z\in U$.
Hence we obtain

$$\left(\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n}{n+p}\right)^{(\alpha-1)t+1}\left(\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}-\frac{n+1}{n+p+1}\right)^{\beta t} = (U_{n,p}(z))^{1-t}(V_{n,p}(z))^{t}. \tag{8}$$

Putting $U(z)=(U_{n,p}(z))^{1-t}(V_{n,p}(z))^{t}$, we get $U(0)=1$ and

$$|\arg\{U(z)\}|\leqq(1-t)|\arg\{U_{n,p}(z)\}|+t|\arg\{V_{n,p}(z)\}|\leqq\frac{\pi}{2}, \tag{9}$$

because $\mathrm{Re}\{U_{n,p}(z)\}>0$ and $\mathrm{Re}\{V_{n,p}(z)\}>0$. This implies that $\mathrm{Re}\{U(z)\}>0$. Thus we obtain that $f(z)\in S_{n,p}((\alpha-1)t+1,\ \beta t)$ for $0\leqq t\leqq 1$.

This completes the proof of the theorem.

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