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LINEAR OPERATORS FOR WHICH $\|T\|=\|\mathrm{Re}T\|$

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0. Introduction.

Let H be a separable, complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. For $T\in B(H)$, let $\sigma(T)$, $\mathrm{Re}T$, and $\mathrm{Im}T$ denote, respectively, the spectrum, the real part, and the imaginary part of T. Note that $\mathrm{Re}T=\dfrac{T+T^*}{2}$ and $\mathrm{Im}T=\dfrac{T-T^*}{2i}$. It is easy to see that $T=\mathrm{Re}T+i\ \mathrm{Im}T$ and T is Hermitian if and only if $\mathrm{Im}T=0$.

It is well known that $\|\mathrm{Re}T\|\leqslant\|T\|$. The existence of skew-Hermitian operators shows that nothing like the reverse inequality could be true. The main purpose of this note is to characterize those operators T for which $\|T\|=\|\mathrm{Re}T\|$. This is accomplished in section 1, in which several interesting corollaries are presented. For convenience we denote by (WH) the class of all operators T satisfying $\|T\|=\|\mathrm{Re}T\|$. (The letters WH suggest "weakly Hermitian"). It is obvious that Hermitian operators are weakly Hermitian, but the converse is not true, even if the underlying Hilbert space is finite dimensional. However, under more restrictive conditions weakly Hermitian operators become Hermitian. In section 2, we consider some algebraic properties of weakly Hermitian operators. Section 3 is devoted to the characterization of those compact operators T for which $\|T\|_p=\|\mathrm{Re}T\|_p$, here $\|\cdot\|_p$ denotes the Schatten p-norm.

1. Weakly Hermitian operators

Our main result can be stated as follows.

THEOREM 1. *For $T\in B(H)$, the following two conditions are equivalent*:
(*a*) $T\in$(WH);
(*b*) *There exists a real number $t\in\sigma(T)$ such that $|t|=\|T\|$.*

PROOF. To see that (b) implies (a) we first notice that $t\in\partial\sigma(T)$ (the boundary of $\sigma(T)$). Hence there exists a sequence of unit vectors $\{x_n\}$ such that $(T-t)\ x_n\longrightarrow 0$ as $n\longrightarrow\infty$. Now

$$\|(T^*-t)x_n\|^2=\|T^*x_n\|^2+t^2-(T^*x_n,\ tx_n)-(tx_n,\ T^*x_n)$$

$$=\|T^{*}x_n\|^2+t^2-t(x_n,\ Tx_n)-t(Tx_n,\ x_n)$$
$$=\|T^{*}x_n\|^2+t^2-t(x_n,\ (T-t)x_n,\ x_n)-t((T-t)x_n,x_n)-2t^2$$
$$=\|T^{*}x_n\|^2-t^2-t(x_n,\ (T-t)x_n)-t((T-t)x_n,\ x_n).$$

Since $\|T^{*}x_n\|^2-t^2\leqslant 0$ and $(T-t)x_n\longrightarrow 0$ as $n\longrightarrow\infty$, it follows that $(T^{*}-t)x_n\longrightarrow 0$ as $n\longrightarrow\infty$. Therefore $(\mathrm{Re}T-t)x_n=\frac{1}{2}((T-t)x_n+(T^{*}-t)x_n)\longrightarrow 0$ as $n\longrightarrow\infty$, and so $t\in\sigma(\mathrm{Re}T)$, from which it follows that $|t|\leqslant\|\mathrm{Re}T\|$. Thus $\|T\|=\|\mathrm{Re}T\|$. For the other implication, choose a real number $t\in\sigma(\mathrm{Re}T)$ such that $|t|=\|T\|=\|\mathrm{Re}T\|$. This is possible since $\mathrm{Re}T$ is Hermitian. Hence there exists a sequence of unit vectors $\{x_n\}$ such that $(\mathrm{Re}T)x_n-tx_n\longrightarrow 0$ as $n\longrightarrow\infty$.

Now $\|(T-t)x_n\|^2=\|Tx_n\|^2+t^2-(Tx_n,\ tx_n)-(tx_n,\ Tx_n)$
$$=\|Tx_n\|^2+t^2-2t((\mathrm{Re}T)x_n,\ x_n)$$
$$\leqslant 2t^2-2t(\mathrm{Re}T)x_n,\ x_n).$$

Since $(\mathrm{Re}T)x_n-tx_n\longrightarrow 0$ as $n\longrightarrow\infty$, it follows that $Tx_n-tx_n\longrightarrow 0$ as $n\longrightarrow\infty$. Therefore $t\in\sigma_{\mathrm{ap}}(T)$ (the approximate point specrum of T) as required.

An immediate consequences of the main result are the following corollaries.

COROLLARY 1. *If $T\in$(WH), then $0\in\sigma(\mathrm{Im}T)$.*

PROOF. Since $T\in$(WH), there exists a real number $t\in\sigma(T)$ such that $|t|=\|T\|$. Therefore a sequence of unit vectors $\{x_n\}$ exists such that $(T-t)x_n\longrightarrow 0$ as $n\longrightarrow\infty$. Thus as in the proof of Theorem 1 we have $(T^{*}-t)x_n\longrightarrow 0$ as $n\longrightarrow\infty$, Hence $(T-T^{*})x_n=(T-t)x_n-(T^{*}-t)x_n\longrightarrow 0$ as $n\longrightarrow\infty$, and so $0\in\sigma(\mathrm{Im}T)$.

COROLLARY 2. *If $T\in$(WH) is quasinilpotent, then $T=0$.*

PROOF. Since $T\in$(WH), there exists a real number $t\in\sigma(T)$ such that $|t|=\|T\|$. But $\sigma(T)=\{0\}$. Thus $\|T\|=0$ and so $T=0$.

COROLLARY 3. *If T is a non-unitary isometry, then $T\in$(WH).*

PROOF. This follows from Theorem 1 and the fact that the spectrum of any non-unitary isometry is the closed unit disc. Corollary 3 generalizes a well known result about the unilateral shift [1]. Weakly Hermitian operators are not necessarily Hermitian. A two-dimensional example is $T=\begin{bmatrix} i & 0\\ 0 & 1\end{bmatrix}$.

However, if $\sigma(T)$ is real and the underlying Hilbert space is two-dimensional, then we have the following positive result.

THEOREM 2. *If H is a two-dimensional Hilbert space and $T\in$(WH), then T*

is normal. Moreover, if $\sigma(T)$ *is real, then* T *is Hermitian.*

POOF. Let $\sigma(T)=\{a, b\}$. By Theorem 1, we may assume that a is real and $|a|=\|T\|$. We will consider two cases.

Case 1. Assume $a\neq b$. Let x and y be unit eigenvectors corresponding to a and b respectively. Since $Tx=ax$ and $|a|=\|T\|$, it follows by an argument similar to that in the proof of Theorem 1, that $T^{*}x=ax$. Hence $b(y, x)=(Ty, x)=(y, T^{*}x)=(y, ax)=a(y, x)$. Thus $(a-b)(y, x)=0$. Since $a\neq b$ we see that $(x, y)=0$, that is x and y are orthogonal. Therefore with respect to the decomposition $H=\langle x\rangle\ominus\langle y\rangle$, T has the matrix representation $T=\begin{bmatrix} a & 0 \\ 0 & b\end{bmatrix}$ and so T is normal.

Case 2. Assume $a=b$. Then Theorem 1 implies $|a|=\|T\|$ and a is real. Let x be a unit eigenevector corresponding to a. Therefore $Tx=ax$ and $T^{*}x=ax$. If y is a unit vector in $\langle x\rangle^{\perp}$, then $Ty=cx+dy$ where $c=(Ty, x)$ and $d=(Ty, y)$. Now $c=(y, T^{*}x)=a(y, x)=0$. Hence $Ty=dy$, and so $d\in\sigma(T)=\{a\}$. Therefore $Ty=ay$ and the proof is complete.

EXAMPLE 1. If H has dimension bigger than two, then there exist operators in (WH) with real spectrum which are not Hermitian. For example, if $T=\begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{bmatrix}$ where a is a real number with $a^2>\frac{3+\sqrt{5}}{2}$, then $\sigma(T)=\{1, a\}$ is real, $|a|=\|T\|$ and so $T\in$(WH), but T is not Hermitian.

EXAMPLE 2. On $L^2(0, 1)$, the space of all square integrable functions on the unit interval, the Volterra operator is defined by $(Vf)(t)=\int_0^t f(s)ds$. It is known [1] that V is quasinilpotent and $V+V^{*}$ is a rank one projection onto the space of constant functions. Let $T=(1+V)^{-1}$, then $\sigma(T)=\{1\}$ and $\|T\|=1$ [1]. Hence according to Theorem 1, $T\in$(WH), yet it is not Hermitian. An alternative proof of the fact that $T\in$(WH) goes like this

$$\begin{aligned} 2\mathrm{Re}T &= (1+V)^{-1}+(1+V^{*})^{-1} \\ &= (1+V)^{-1}(2+V+V^{*})(1+V^{*})^{-1} \\ &\geqslant 2(1+V)^{-1}(1+V^{*})^{-1} \\ &= 2TT^{*}. \end{aligned}$$

Therefore $\|\mathrm{Re}T\|\geqslant\|T^{*}\|^2=1$ and so $\|\mathrm{Re}T\|=\|T\|=1$.

For finite dimensional Hilbert spaces, the situation is different as indicated by the next theorem.

THEOREM 3. *Let H be a finite dimensional Hilbert space. If $T\in$(WH) with $\sigma(T)=\{a\}$, then $T=a$.*

PROOF. By Theorem 1, a must be real and $|a|=\|T\|$. Let n be the dimension of H. The theorem is ceartinly true for $n=1$. Assume the theorem hold for Hilbert spaces of dimension less than n. Let x be a unit eigenvector corresponding to a. Then $Tx=ax$ together with $|a|=\|T\|$ imply $T^*x=ax$. Thus $\langle x\rangle$ is a reducing subspace for T. With respect to the decomposition $H=\langle x\rangle\oplus\langle x\rangle^{\perp}$, T has the matrix representation $T=\begin{bmatrix} a & 0 \\ 0 & S\end{bmatrix}$ for some operator S acting on the $n-1$ dimensional Hilbert space $\langle x\rangle^{\perp}$. Now $\sigma(T)=\{a\}\cup\sigma(S)$, and so $\sigma(S)=\{a\}$, from which it follows that $|a|\leqq\|S\|$. But $|a|=\|T\|=\max\{|a|, \|S\|\}$ implies $\|S\|\leqq|a|$. Hence $\|S\|=|a|$, and so by Theorem 1, we have $S\in$(WH). The induction assumption now implies that $S=a$ on $\langle x\rangle^{\perp}$. Therefore $T=a$ on H.

2. Some algebraic properties.

We will utilize Theorem 1 to investigate some algebraic properties of weakly Hermitian operators. It is trivial to notice that (WH) is closed under conjugation, that is if $T\in$(WH), then $T^*\in$(WH). Unfortunately (WH) is not closed under addition multiplication, and inversion. To demonstrate this we consider the following examples.

EXAMPLE 3. Let $T=\begin{bmatrix} i & 0 \\ 0 & 1\end{bmatrix}$, $S=\begin{bmatrix} 1 & 0 \\ 0 & i\end{bmatrix}$. Then T, $S\in$(WH) and $TS=ST$, yet $T+S\notin$(WH) and $TS\notin$(WH).

To see that (WH) is not closed under inversion we appeal to Example 2. For if $T=(1+V)^{-1}$, then $T\in$(WH) while $T^{-1}=1+V$ is not in (WH). This is because $\|\mathrm{Re}T^{-1}\|=\|1+\mathrm{Re}V\|=3/2$. (Recall that $V+V^*$ is a rank one projection). If $T^{-1}=1+V$ were in (WH), then by Theorem 1, there would exist a real number $t\in\sigma(T^{-1})$ such that $|t|=3/2$. But $\sigma(T^{-1})=\{1\}$ and so T^{-1} cannot be in (WH). A less informative example is $\begin{bmatrix} 2 & 0 \\ 0 & i\end{bmatrix}$.

Example 3 shows that the product of commuting operators in (WH) need not be in (WH). However, the next result shows that the power of every operator in (WH) is again in (WH).

THEOREM 4. *If* $T\in$(WH), *then* $T^n\in$(WH) for every $n>1$.

PROOF. Assume $T\in$(WH). Then there exists a real number $t\in\sigma(T)$ such that $|t|=\|T\|$. By the spectral mapping theorem, $t^n\in\sigma(T^n)$. Therefore $\|T\|^n=|t^n|\leqslant\|T^n\|\leqslant T\|^n$, and so $|t^n|=\|T^n\|$, from which it follows by Theorem 1 that $T^n\in$(WH).

The above result cannot be extended to general polynomials as demonstrated by the following example.

EXAMPLE 4. Let $T=\begin{bmatrix}-1 & 0\\ 0 & i\end{bmatrix}$ and $p(t)=t+t^2$. Then it is clear that $T\in$(WH). On the other hand $p(T)=\begin{bmatrix}0 & 0\\ 0 & i-1\end{bmatrix}\notin$(WH).

We would like to end this section by remarking that (WH) is not translation invariant, that is if $T\in$(WH) and λ is a scalar, then $T-\lambda$ is not necessarily in (WH). This can be easily verified by employing 2×2 matrices.

3. Operators for which $\|T\|_p=\|\mathrm{Re}T\|_p$.

Motivated by the results above and by the argument which follows Theorem 1.22 in [2], we now consider those compact operators for which $\|T\|_p=\|\mathrm{Re}T\|_p$. For any compact operator T, let $|T|$ denote the positive square root of T^*T, and let $s_1(T), s_2(T), \cdots$ be the eigenvalues of $|T|$ in nonincreasing order repeated according to multiplicity. If, for some $p\geqslant1$, we have $\sum s_j(T)^p<\infty$, then we say that T belongs to the Schatten p-class C_p which is normed with $\|T\|_p=(\sum s_j(T)^p)^{1/p}$. We refere to [2] for the general theory of the Schatten p-classes. Utilizing the remarkable Clarkson-McCarthy inequalities [2] enables us to characterize those compact operators T in C_p for which $\|T\|_p=\|\mathrm{Re}T\|_p$ $(p>1)$. In fact, it is shown that $\|T\|_p=\|\mathrm{Re}T\|_p$ if and only if T is Hermitian.

For the reader's convenience the Clarkson-McCarthy inequalities are included.

THEOREM 5. *Let A and B be operators on H. Then*

(*a*) $\|A+B\|_p^p+\|A-B\|_p^p\leqslant2^{p-1}(\|A\|_p^p+\|B\|_p^p)(2\leqslant p<\infty)$

(*b*) $\|A+B\|_p^q+\|A-B\|_p^q\leqslant2(\|A\|_p^p+\|B\|_p^p)^{q/p}$ $(1<p\leqslant2$ *and* $1/p+1/q=1)$

THEOREM 6. *Let T be in* C_p $(1<p<\infty)$. *Then* $\|T\|_p=\|\mathrm{Re}T\|_p$ *if and only if T is Hermitian.*

PROOF. First assume $2\leqslant p<\infty$. Then by Clarkson-McCarthy inequality (*a*) applied to T and T^* we have

$\|T+T^*\|_p^p+\|T-T^*\|_p^p \leq 2^{p-1}(\|T\|_p^p+\|T^*\|_p^p)$. Thus $2^p(\|\mathrm{Re}T\|_p^p+\|\mathrm{Im}T\|_p^p) \leq 2^p\|T\|_p^p$, and so $\|\mathrm{Re}T\|_p^p+\|\mathrm{Im}T\|_p^p \leq \|T\|_p^p$. If $\|T\|_p=\|\mathrm{Re}T\|_p$, then $\mathrm{Im}T=0$. Hence T is Hermitian. Next assume $1<p\leq 2$. Applying inequality (*b*) to T and T^* we get $2^q(\|\mathrm{Re}T\|_p^q+\|\mathrm{Im}T\|_p^q) \leq 2^{1+q/p}\|T\|_p^q$, where $1/q+1/p=1$. Thus $\|\mathrm{Re}T\|_p^q+\|\mathrm{Im}T\|_p^q \leq \|T\|_p^q$. If $\|T\|_p=\|\mathrm{Re}T\|_p$, then $\mathrm{Im}T=0$, and so T is Hermitian as required.

For $p=1$, the above characterization is not applied. For let $T=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+i\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$. Then $\|T\|_1=\|\mathrm{Re}T\|_1=2$, yet T is not Hermitian [2]. However, if we assume that $\mathrm{Re}T$ and $\mathrm{Im}T$ commute, that is, T is normal, then the spectral theorem for compact normal operators gives us the following result.

THEOREM 7. *Let $T\in C_1$ be a normal operator for which $\|T\|_1=\|\mathrm{Re}T\|_1$ Then T is Hermitian.*

PROOF. By the spectral theorem for compact normal operators, we may assume without loss of generality that T is a diagonal operator. Let $T=\mathrm{Diag}(t_j)$. Now $\|T\|_1=\Sigma|t_j|$ and $\|\mathrm{Re}T\|_1=\Sigma|\mathrm{Re}t_j|$. If $\|T\|_1=\|\mathrm{Re}T\|_1$, then $\Sigma(|t_j|-|\mathrm{Re}t_j|)=0$. But $|\mathrm{Re}t_j|\leq|t_j|$ for all j. Thus $|t_j|=|\mathrm{Re}t_j|$ for all j, and so $\mathrm{Im}t_j=0$ for all j. Therefore $\mathrm{Im}T=0$ as required.

We conclude this note by raising the following question.

QUESTION. What is a reasonable characterization for C_1 operators T for which $\|T\|_1=\|\mathrm{Re}T\|_1$?

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