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ON INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN-REID TYPE II

By A. A. S. Zahrout and Z. M. Aly

Abstract: In this paper we present several nonlinear generalization of Gronwall-Bellman-Reid inequalities. These results provided genuine upper estimates.

1. Introduction

The celebrated Gronwall-Bellman-Reid inequality [2, 3] and its variants play a vital role in the study of the stability and boundedness properties of differential and integral equations. The aim of this paper is to establish more integral inequalities which can be used as a tool in obtaining the lower bounds for the solutions of differential and integral equations.

We make the following assumptions in our subsequent discussions.

(A_1) $n(t)$, $f(t)$, $g(t)$ and $h(t)$ are real-valued, positive, continuous functions defined on $I=[0,\infty]$.

(A_2) $n(t)$ is a positive, monotonic, nondecreasing continuous function defined on I.

(A_3) A function Φ is said to belong to a class S if

(i) $\Phi(u)$ is positive, nondecreasing and continuous on I,

(ii) $\Phi(u)/v \leqslant \Phi(u/v)$ for $u>0$, $v\geqslant 1$.

(A_4) $W(u)$ and $H(u)$ are real-valued, positive, continuous, monotonic, nondecreasing, subadditive, and submultiplicative functions for $u>0$, $W(0)=0$, $H(0)=0$.

(A) $\Psi(t)$ is a positive, nondecreasing and continuous function on I and $\Psi(0)=0$.

(A) $P(t)>0$ is nonnegative, continuous, nonderessing function on I.

We shall make use of the following theorem (Pachpatte [5]):

Theorem A: *Let* (A_1) *and* (A_2) *be true. Suppose further that*

$$x(t)\leqslant n(t)+g(t)\Big(\int_0^t f(s)\ x(s)\ ds\Big),\ t\in I.$$

Then

$$x(t)\leqslant n(t)\,[1+g(t)\Big(\int_0^t f(s)\ \exp\ \Big(\int_s^t g(r)f(r)dr\Big)ds)],\ t\in I.$$

2. Main results

We wish to establish more general integral inequalities which can be used in applications.

THEOREM 1. *Let* (A_1), (A_4), (A_5) *and* (A_6) *be true. Suppose further that the inequality.*

$$x(t) \leqslant P(t) + g(t)\Big(\int_0^t f(s)x(s)ds\Big) + \Psi\Big(\int_0^t h(s)W(x(s))ds\Big) \tag{2.1}$$

is satisfied for all $t \in I$. *Then*

$$x(t) \leqslant [P(t) + \Psi(G^{-1}[G\Big(\int_0^t h(s)W(P(s)k(s)ds) + \int_0^t h(s)W(k(s))ds]\Big)]k(t), \tag{2.2}$$

where

$$k(t) = \Big[1 + g(t)\Big(\int_0^t f(s) \exp\Big(\int_s^t g(r)f(r)dr)ds\Big)\Big],$$

$$G(r) = \int_{r_0}^r [1/W(\Psi(s))]\, ds, \qquad 0 < r_0 \leqslant r \tag{2.3}$$

and G^{-1} *is the inverse of* G, *and* $t \leqslant I_0 = [0,\ b]$

$$I_0 = \Big\{t \in I \mid G\Big(\int_0^t h(s)k(s)\Big) + \int_0^t h(s)W(k(s))ds \in \mathrm{Dom}\,(G^{-1})\Big\}.$$

PROOF. Define

$$n(t) = p(t) + \Psi\int_0^t h(s)W(x(s))ds). \tag{2.4}$$

Then inequality (2.1) can be written in the form

$$x(t) \leqslant n(t) + g(t)(f(s)x(s)ds), \text{ for all } t \in I. \tag{2.5}$$

Then applying theorem A we have

$$x(t) \leqslant n(t)\Big[1 + g(t)\Big(\int_0^t f(s) \exp\Big(\int_s^t g(r)f(r)dr\Big)ds\Big)\Big]$$

$$\text{i.e. } x(t) \leqslant n(t)k(t) \tag{2.6}$$

Let

$$v(t) = \int_0^t h(s)W(x(s))ds, \qquad v(0) = 0$$

Then equation (2.6) takes the form

$$n(t) = p(t) + \Psi(v(t)). \tag{2.7}$$

Hence

$$x(t) \leqslant [P(t) + \Psi(v(t))]\,k(t) \tag{2.8}$$

i. e.

$$x(t) \leqslant p(t)k(t) + \Psi(v(t))k(t)$$

Consequently

$$W(x(t)) \leqslant W(p(t)k(t)) + W(\Psi(v(t))W(k(t)),$$

since W is submultiuplicative. Hence

$$\frac{h(t)W(x(t))}{W(\Psi(v(t)))} \leqslant \frac{h(t)}{W(\Psi(v(t)))} W(P(t)k(t) + h(t)W(k(t))$$

Because of equation (2.3), this reduces to

$$\frac{d}{dt} G(v(t) \leqslant \frac{h(t)}{W\Psi(v(t))} W(p(t).\ k(t)) + h(t)W(k(t).$$

Now integrating from 0 to t, we obtain

$$G(v(t)) - G(v(0)) \leqslant \int_0^t \frac{h(s)}{W(\Psi(v(s)))} W(p(s)k(s)) + \int_0^t h(s)W(k(s))ds.$$

Since $G(v(0))=0$, hence

$$v(t) \leqslant G^{-1}\Big\{G\Big(\int_0^t h(s)W(p(s)k(s))ds\Big) + \int_0^t h(s)W(k(s))ds\Big\}. \tag{2.9}$$

using (2.7), (2.8) and (2.9), we obtain the desired bound in (2.2). This completes the proof of the theorem.

Now we shall use the following theorem by Pachpatte [5]

THEOREM B (Pachpatte [5]). *Let* (A_1), (A_2), (A_3) *and* (A_4) *be true. Suppose that the inequality*

$$x(t) \leqslant n(t) + H^{-1}\Big[\Phi\Big(\int_0^t f(s)H(x(s))ds\Big)\Big],\ t \in I$$

holds. Then $\quad x(t) \leqslant n(t)k(t)$

where

$$k(t) = H^{-1}\Big[1 + \Phi\Big(G^{-1}\Big[G(0) + \int_0^t f(s)ds\Big]\Big)\Big],\ t \in I_0, \tag{2.10}$$

$$G(r) = \int_{r_0}^r [1/1 + \Phi(s)]\,ds, \qquad 0 < r_0 \leqslant r$$

THEOREM 2. *Let* (A_1), (A_3) *and* (A_4) *be true, Suppose further that the inequality*

$$x(t) \leqslant x_0 + H^{-1}\Big[\Phi\Big(\int_0^t f(s)H(x(s)))ds\Big] + \int_0^t g(s)W(x(s))ds \tag{2.11}$$

is satisfied for all $t \in I$, *where* x_2 *is a positive constant. Then*

$$x(t)\leq\Omega^{-1}\Big[\Omega(x_0)+\int_0^t g(s)W(k(s))\Big](k(t) \qquad t\in I_0 \tag{2.12}$$

where

$$\Omega(r)=\int_{r_0}^{r}[1/W(s)]\,ds \qquad 0<r_0<r \tag{2.13}$$

and Ω^{-1} *is the inverse function of* Ω *and* $t\in I_0$

$$I_0=\Big\{t\in I\,|\,\Omega(x_0)+\int_0^t g(s)W(k(s))ds\in\mathrm{Dom}(\Omega^{-1})\Big\}.$$

PROOF. Define

$$n(t)=x_0+\int_0^t g(s)W(x(s))ds.$$

Hence inequality (2.11) takes the form

$$x(t)\leq n(t)+H^{-1}\Big[\Phi\Big(\int_s^t f(s)H(x(s))ds\Big)\Big]. \tag{2.14}$$

Since $n(t)$ be a positive, monotonic, nondecreasing, continuous function defined on I, and using (A_4) and theorem B, we have

$$x(t)\leq n(t)k(t). \tag{2.15}$$

Since W is submultiplicative,

$$\frac{g(t)W(x(t))}{W(n(t))}\leq g(t)W(k(t)).$$

Using the definition of Ω by (2.13), this reduces to

$$\frac{d}{dt}\Omega(n(t))\leq g(t)W(k(t)).$$

Integrating from 0 to t we obtain

$$\Omega(n(t))-\Omega(x_0)\leq\int_0^t g(s)W(k(s))ds.$$

$$n(t)\leq\Omega^{-1}\Big[(x_0)+\int_0^t g(s)W(K(s))ds\Big]. \tag{2.16}$$

The desired bound in (2.12) follows directly from (2.15) and (2.16).
This completes the proof of the theorem.

THEOREM 3. *Let* (A_1) (A_3) (A_4), (A_5) *and* (A_6) *be true. Suppose further that the inequality.*

$$x(t)\leq p(t)+H^{-1}\Big[\Phi\Big(\int_0^t f(s)H(x(s))ds\Big)\Big]+\Psi\Big(\int_0^t g(s)W(x(s))ds\Big) \tag{2.17}$$

is satisfied for all $t\in I$. *Then*

$$x(t) \leqslant \Big[p(t)+\Psi(\Omega^{-1})\Big[\Omega\Big(\int_0^t g(s).\ W(P(s)k(s))\Big)ds+$$
$$\int_0^t g(s)W(k(s))ds\Big]\Big)\Big]k(t) \qquad t\in I_0 \tag{2.18}$$

where k is defined by (2.10) *and* Ω *is defined by*

$$\Omega(r)=\int_{r_0}^{r}[1/W(\Psi(s))]\,ds \qquad 0<r_0<r \tag{2.19}$$

and Ω^{-1} *is the inverse function of* Ω, $t\in I_0$,

$$I_0=\Big\{t\in I\,|\,\Omega\Big(\int_0^t g(s)W(p(s)k(s))ds\Big)+$$
$$\int_0^t g(s)W(k(s))ds\in \mathrm{Dom}(\Omega^{-1})\Big\}.$$

PROOF. Define.

$$n(t)=p(t)+\Psi\Big(\int_0^t g(s)W(x(s))ds\Big). \tag{2.20}$$

Then the inequality (2.17) reduces to

$$x(t)\leqslant n(t)+H^{-1}\Big[\Phi\Big(\int_0^t f(s)H(x(s(s))ds\Big].$$

Applying theorem B, we obtain.

$$x(t)\leqslant n(t)k(t) \tag{2.21}$$

Let $\qquad v(t)=\int_0^t g(s)W(x))ds,\ \ v(0)=0.$

Now from (2.20) and (2.21) we have

$$x(t)\leqslant[p(t)+\Psi(v(t))]\,k(t)). \tag{2.21}$$

Thus

$$W(x(t))\leqslant W(p(t)k(t))+W(\Psi(v(t)k(t))).$$

since W is submultiplicative. Hence

$$\frac{g(t)W(x(t))}{W(\Psi(v(t)))}\leqslant\frac{g(t)}{W(\Psi(v(t)))}W(p(t)k(t))+g(t)W(k(t)).$$

Using the definition of Ω by (2.19), we have

$$\frac{d}{dt}\Omega(v(t)\leqslant\frac{g(t)}{W(\Psi(v(t)))}W(p(t)k(t))+g(t)W(k(t).$$

Integrating from 0 to t, we get

$$\Omega(v(t))-\Omega(v(0))\leqslant\int_0^t\frac{g(s)}{W(\Psi(v(s)))}W(P(s)k(s))ds+\int_0^t g(s)W(k(s)ds, \tag{2.23}$$

The desired bound in (2.18) follows directly from (2.21), (2.22) and (2.23). This completes the proof of the theorem.

Now, we establish the following more general integral inequality which may be convinient in some applications.

THEOREM 4. *Let* (A_1), (A_3), (A_4), (A_5) *and* (A_6) *be true. Let* $W(t, u)$ *be a positive, continuous, monotonic nondecreasing function in* u *for fixed* t, $u>0$. *Suppose further more that the inequality.*

$$x(t)<p(t)+H^{-1}\Big[\Phi\Big(\int_0^t f(s)H(x(s))ds\Big)\Big]$$

$$+h(t)\Psi\Big(\int_0^t q(s)W(s,\ x(s))ds\Big), \tag{2.24}$$

is satisfied for all $t\in I$. *Then*

$$x(t)<[p(t)+h(t)\Psi(r(t)]k(t)),\ t\in I_0 \tag{2.25}$$

where $k(t)$ *is defined by* (2.10) *and* $r(t)$ *is the maximal solution of*

$$r(t)=g(t)W(t,k(t))[p(t)+h(t)\Psi(r(t))],\ r(0)=0, \tag{2.26}$$

existing on I.

PROOF. Define

$$n(t)=p(t)+h(t)\Psi\Big(\int_0^t q(s)W(s,\ x(s))ds\Big). \tag{2.27}$$

Hence the inequality (2.24) reduces to

$$x(t)<n(t)+H^{-1}\Big[\Phi\Big(\int_0^t g(s)H(x(s))ds\Big)\Big].$$

Since $n(t)$ is positive, monotonic, nondecreasing on I, and making use of theorem B, we have

$$x(t)<n(t)k(t), \qquad t\in I_0 \tag{2.28}$$

Form (2.27) and (2.28) we have

$$x(t)<k(t)[p(t)+h(t)\Psi(v(t))], \tag{2.29}$$

where

$$v(t)=\int_0^t q(s)W(s,\ x(s))ds, \qquad v(0)=0.$$

Consequently, it follows that

$$v'(t)<q(t)W(t,\ k(t))[p(t)+h(t)\Psi(v(t))]. \tag{2.30}$$

A suitable application of theorem 1.4.1 given in [4] to (2.30) and (2.26) yields

$$v(t) \leqslant r(t) \tag{2.31}$$

where $r(t)$ is the maximal solution of (2.26) such that $r(0)=v(0)=0$.
The desired bound in (2.25) follows (2.29) and (2.31). This completes the proof of the theorem.

REMARKS. (1) Similarly we can find upper bounds for the inequalities of the form

$$x(t) \leqslant n(t) + \Phi\Big(\int_0^t f(s)H(x(s))ds\Big), \qquad t \in I.$$

(2) Modified theorems can be obtained if we change the definition of $G(t)$ to

$$G(r) = \int_{r_0}^{r} [1/H(1+\Phi(s))]\, ds, \qquad 0 < r_0 \leqslant r.$$

Now we shall need the following theorem to prove theorem 5.

THEOREM C(Pachpatte [6]).
Let (A_1) *and* (A_2) *be true. Suppose further more that the inequality*

$$x(t) \leqslant n(t) + \int_0^t f(s)x(s)ds + \int_0^t f(s)\Big(\int_0^s g(r)x(r)dr\Big)ds, \ t \in I$$

holds. Then

$$x(t) \leqslant n(t)\Big(1 + \int_0^t f(s) \exp\Big(\int_0^s (f(r)+g(r))dr\Big)ds\Big) \ t \in I.$$

THEOREM 5. *Let* (A_1), (A_4), (A_5) *and* (A_6) *be true. Suppose more that the inequality.*

$$x(t) \leqslant p(t) + \int_0^t f(s)x(s)ds + \int_0^t f(s)\Big(\int_0^s g(r)x(r)dr\Big)ds + \Psi\Big(\int_0^t h(s)W(x(s))ds\Big) \tag{2.32}$$

is satisfied for all $t \in I$. *Then*

$$x(t) \leqslant \Big[p(t) + \Psi(G^{-1}\Big[G\Big(\int_0^t h(s)W(p(s)k(s))ds + \int_0^t h(s)W(k(s))\Big]\Big)\Big].\ k(t), \qquad t \in I_0 \tag{2.33}$$

where $G(r)$ *is defined by* (2.3) *and*

$$k(t) = \Big[1 + \int_0^t f(s) \exp\Big(\int_0^s (f(r)+g(r))dr\Big)ds\Big]. \tag{2.34}$$

PROOF. Define

$$n(t)=P(t)+\Psi\Big(\int_0^t h(s)W(x(s))ds\Big). \tag{2.35}$$

Therefore, using Theorem C, the inequality (2.32) can be written as

$$x(t)\leqslant n(t)\Big[1+\int_0^t f(s)\ \exp\ \Big(\int_0^s (f(r)+g(r)dr)ds\Big],\ t\in I,$$

i. e.

$$x(t)\leqslant n(t)k(t). \tag{2.36}$$

Equation (2.34) can be reduced to

$$n(t)=p(t)+\Psi(v(t) \tag{2.37}$$

where

$$v(t)=\int_0^t h(s)W(x(s))ds.$$

Further more, since W is submultiplicative,

$$W(x(t))\leqslant W(n(t))W(k(t))$$

Hence using (2.37), we have

$$h(t)W(x(t))\leqslant h(t)W(p(t)k(t))+h(t)W(\Psi(v(t)))W(k(t)).$$

Thus

$$\frac{h(t)W(x(t))}{W(\Psi(v(t)))}=\frac{v'(t)}{W(\Psi(v(t)))}=\frac{d}{dt}G(v(t))$$
$$\leqslant\frac{h(t)W(p(t)k(t))}{W(\Psi(v(t)))}+h(t)W(k(t)).$$

Integrating from 0 to t, we have

$$G(v(t))-G(v(0))\leqslant\int_0^t h(s)W(k(t))ds$$
$$+\int_0^t\frac{h(s)W(p(s)k(s))}{W(\Psi(v(s)))}ds.$$

Hence

$$v(t)\leqslant G^{-1}\Big\{\int_0^t h(s)W(k(t))ds+G\Big(\int_0^t h(s)W(p(s)k(s))ds\Big)\Big\}.$$

Thus (2.37) can be written as

$$n(t)\leqslant p(t)+\Psi(G^{-1}\Big\{\int_0^t h(s)W(k(s))ds$$
$$+G\Big(\int_0^t h(s)W(p(s)k(s))ds\Big)\Big\}. \tag{2.38}$$

The desired bound follows from (2.30) and (2.38). This completes the proof of the theorem.

REMARK 3. Similary we can find upper bounds for the inequalities of the form

$$x(t) \leqslant p(t) + \int_0^t g(s)(x(s) + \int_0^s g(r)\Phi(x(r))dr)ds$$
$$+ \Psi\Big(\int_0^t h(s)W(x(s))ds\Big).$$

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Department of Mathematics,
Faculty of Science,
Al-Azhar University,
Nasr-City, Cairo,
Egypt.