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ON INTEGRAL INEQUALITIES OF GRONWALL-BELLMAN-REID TYPE I

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Abstract: In this paper we wish to establish some new integral in equalities of the Gronwall-Bellman-Reid type that have a wide range of applications in the differential and integral equations.

1. Introduction

It is well recognised that the integral in equalities furnish a very general comparison principle in studying many qualitative as well as quantitative properties of solutions of differential equations. The celebrated Gronwall inequality known as Gronwall-Bellman-Reid inequality [2, 3, 8] provides explicit bounds on solutions of a class of linear integral inequalities. On the basis of various motivations this inequality has been extended and used in various contexts. The first nonlinear version of this inequality is due to Behari [3] which has been further generalized in several different directions. An extensive survey of these generalizations is given recently by Beesack [1], Deo et al [5, 6, 7], and chu [4]. Also applications of Gronwall-Reid inequality to the boundedness and stability problems of volterra integral equations and integro-differential equations of more general type will be discussed latter [12].

We shall make use the following theorem by Pachpatte,

THEOREM A (Pachpatte [10]).

Let $x(t, f(t)$ and $g(t)$ be real valued nonnegative continuous functions defined on I, and $n(t)$ be positive, monotonic, nondecreasing continuous function defened on $I=[0,\infty)$, for which the inequality

$$x(t)\leqslant n(t)+\int_0^t f(s)x(s)ds+\int_0^t g(s)n^{[1-p]}(s)x^p(s)ds \tag{1.1}$$

holds for each $t\in I$, $0\leqslant p\leqslant 1$. Then

$$x(t)\leqslant n(t)k_1(t) \tag{1.2}$$

Where

$$k_1(t)=\exp\Big(\int_0^t f(s)ds\Big)\Big[1+(1-p)\int_0^t g(s)\exp\Big(1-(1-p)\int_0^s f(r)dr\Big)ds\Big]^{1-p} \tag{1.3}$$

for all $t \in I$.

2. Main results

The attractive feature of the integral inequalities presented in this paper is that, we are taking full advantage of the monotonicity condition on a function $n(t)$ (defined later) to establish some new integral inequalities. However, if we drop the monotonicity condition on this function $n(t)$ then it is not so easy to establish the bounds for the integral inequalities considered in this paper.

An interesting and useful generalizations of Gronwall-Bellman-Reid inequality is embodied in the following theorem.

THEOREM 1. *Let* $u(t)$, $v(t)$, $w(t)$ *and* $f(t)$ *be real-valued nonnegative continuous (or pieceuise continuous) functious defined on a real* t *interval* $I = [0, \infty)$. *Suppose further for all* $t \in I$, *that*

$$u(t) \leqslant u_0 + \int_0^t (v(s)u(s) + w(s))ds + \int_0^t v(s)\left(\int_0^s f(r)u(r)dr\right)ds \tag{2.1}$$

where u_0 *is a nonnegative constant. Then*

$$u(t) \leqslant u_0 + \int_0^t \Big[w(s) + v(s)\Big\{\exp\Big(\int_0^s v(r) + f(r)dr\Big)\Big(u_0 + \int_0^s w(r) \cdot \exp\Big(-\int_0^r (v(t) + f(t)dt)dr\Big)\Big\}\Big]ds, \tag{2.2}$$

for all $t \in I$.

PROOF. Define a function $R(t)$ such that

$$R(t) = u_0 + \int_0^t (v(s)u(s) + w(s))ds + \int_0^t v(s)\left(\int_0^s f(r)u(r)dr\right)ds, \tag{2.3}$$

Then

$$u(t) \leqslant R(t), \qquad R(0) = u_0$$

Hence we have

$$R(t) = v(t)u(t) + w(t) + v(t)\int_0^t f(r)u(r)dr.$$

Using (2.1) and (2.3) we have:

$$R'(t) \leqslant w(t) + v(t)\Big[R(t) + \int_0^t f(r)R(r)dr\Big]. \tag{2.4}$$

If we put

$$m(t) = R(t) + \int_0^t f(r)R(r)dr, \tag{2.5}$$

with $m(0) = R(0) = u_0$.

Hence

$$R'(t) \leqslant w(t) + v(t)m(t), \qquad R(t) \leqslant m(t)$$

Thus we can write

$$m'(t) - (v(t) + f(t))m(t) \leqslant w(t) \tag{2.6}$$

Integrating (2.6) from 0 to t we obtain

$$m(t) \leqslant \exp\int_0^t (v(s)+f(s))ds \Big[\int_0^t \exp\Big(-\int_0^s v(r)+f(r)\Big)dr w(s)ds + u_0\Big].$$

Then we have

$$R'(t) \leqslant w(t) + v(t)\exp\int_0^t (v(s)+f(s))ds.\ \Big[\int_0^t \exp\Big\{-\int_0^s (v(r)+g(r))dr\Big\}.$$
$$w(s)ds + u_0]. \tag{2.7}$$

Integrating both sides of (2.7) from 0 to t and substituting the values of $R(t)$ in (2.1) we obtain the desired bound in (2.2). Thus completes the proof of the theorem.

We shall prove a more generalization of Gronwall Bellman Reid inequality which can be used in applications.

THEOREM 2. *Let $u(t)$, $v(t)$, $f(t)$ and $g(t)$ be real valued continuous functions defined on I. Suppose for all $t \in I$, that*

$$u(t) \leqslant u_0 + \int_0^t v(s)\,[u(s) + \int_0^s v(r)\Big\{\int_0^r (f(t)u(t) + g(t)u^p(t))dt\Big\}dr]\,ds, \tag{2.8}$$

where $o \leqslant p < 1$, u_0 is a nonnegative constant.

Then

$$u(t) \leqslant u_0 + \int_0^t v(s)\Big[u_0 + \int_s^s v(r)\exp\Big(\int_0^r (v(t)+f(t))dt\Big).$$
$$\Big\{u_0^{1-p} + (1-p)\int_0^r g(t)\exp\Big(-(1-p)\int_0^t (v(k)+f(k))dk)dt\Big)^{1/1-p}\Big\}dr]\,ds, \tag{2.9}$$

for all $t \in I$.

PROOF. Define a function $R(t)$ such that

$$R(t) = u_0 + \int_0^t v(s)\Big[u(s) + \int_0^s v(r)\Big\{\int_0^r (f(t)u(t) + g(t)u^p(t))dt\Big\}dr\Big]ds.$$

Hence

$$u(t) \leqslant R(t), \qquad R(0) = u_0 \tag{2.10}$$

Defferentiating $R(t)$ with respect to t and using (2.10) we have

$$R'(t) \leqslant v(t)\Big[R(t) + \int_0^t v(r)\Big\{\int_0^r (f(t)R(t) + g(t)R^p(t))dt\Big\}dr\Big],$$

Define the function $m(t)$ by

$$m(t)=R(t)+\int_0^t v(r)\left\{\int_0^r [f(t)R(t)+g(t)R(t)]\,dt\right\}dr, \tag{2.11}$$

where $m(0)=R(0)=u_0$.

Differentiating (2.11) with respect to t and using the fact $m(t)\geqslant R(t)$, we have

$$m'(t)\leqslant v(t)m(t)+v(t)\int_0^t (f(r)m(r)+g(r)m^p(r))dr$$

$$\leqslant v(t)\,[m(t)+\int_0^t (f(r)m(r)+g(r)m^p(r)dr]. \tag{2.12}$$

Define the function $n(t)$ dy

$$n(t)=m(t)+\int_0^t f(r)m(r)+g(r)m^p(r)dr, \tag{2.13}$$

where $n(0)=m(0)=R(0)=u_0, m(t)\leqslant n(t)$.

Hence in vertue of (2.13) inequatity (2.12)can be written as

$$m'(t)\leqslant v(t)n(t). \tag{2.14}$$

Differentiating (2.15) and using (2.14) we get

$$n'(t)-(v(t)+f(t))n(t)\leqslant g(t)n^p(t), \qquad 0\leqslant p<1.$$

After simple manipulation we have the estimate

$$n(t)\leqslant\exp\int_0^t (v(r)+f(r))dr.\left[u_0^{1-p}+\int_0^t (1-p)\exp\left\{-(1-p)\int_0^s (v(r)+f(r))dr.\right.\right.$$
$$\left.h(s)ds\right]^{1/1-p}$$

Inequality (2.14) can be written as

$$m'(t)\leqslant v(t)n(t)\leqslant v(t)\Big[u_0+\int_0^t v(s)\exp\int_0^s [v(r)+f(r)]\,dr.$$

$$[u_0^{1-p}+\int_0^s (1-p)\exp(-(1-p)\int_0^r (v(t)+f(t))dt)h(r)dr]^{1/1-p}ds. \tag{2.15}$$

Integrating both sides of (2.15) from 0 to t and using (2.8) we obtain the desired bound (2.9). This completes the proof of the theorem.

Now, we shall apply theorem A, to establish the following more general integral inequalities.

Before stating the theorems, we shall suppose the following

(i) Let $w(u)$ be a positive, continuous, monotonic nondecreasing subadditive and submultiplicative function for $u>0$, $W(0)=0$;

(ii) the function $\Phi(t)\geqslant 0$ be nondecreasing in t and continuous on I, $\Phi(0)=0$,

(iii) The function $m(t)$ defined by

$$m(t)=v(t)+\int_0^t g(r)v^p(r)dr, \quad m(0)=(0)=1 \tag{2.16}$$

(iv) The function G defined by

$$G(r)=\int_{r_0}^{r}[1/W(\Phi(s))]ds, \qquad 0<r_1\leqslant r \tag{2.17}$$

and G^{-1} is the inverse of G, and t is in the subinterval $[0, b]$ of I such that

$$G\Big(\int_0^t h(s)\ W(m(s))\ W(k_1(s))ds\Big)+\int_0^t h(s)W(k_1(s))ds\in \mathrm{Dom}(G^{-1}),$$

where k_1 is defined as (1.3).

THFORFM 3. *Let $x(t)$, $f(t)$, and $h(t)$ be real valued positive continuous function defined on I; $W(u)$, $m(t)$, and $\Phi(t)$ as defined above. Suppose further that the inequality*

$$x(t)\leqslant m(t)+\Phi\Big(\int_0^t k_1(s)W(x(s))ds+\int_0^t f(s)x(s)ds+\int_0^t g(s)\cdot[m(s)$$

$$+\Phi\Big(\int_0^s h(r)W(x(r))dr]^{1-p}x^p(s)ds, \tag{2.18}$$

holds. Then

$$x(t)\leqslant k_1(t)[m(t)+\Phi(G^{-1}[G\Big(\int_0^t h(s)W(m(s))W(k_1(s))ds)$$

$$+\int_0^t h(s)W(k_1(s))ds])] \tag{2.19}$$

for all $t\in I$ and k_1 is defined by (1.3).

PROOF. Define

$$n(t)=m(t)+\Phi\Big(\int_0^t h(s)W(x(s))ds\Big). \tag{2.20}$$

Then inequality (2.18) restated as

$$x(t)\leqslant n(t)+\int_0^t f(s)x(s)ds+\int_0^t g(s)n^{1-p}(s)x^p(s)ds, \tag{2.21}$$

Since $n(t)$ is positive, monotonic, nondecreasing on I, and applying theorem A, we have

$$x(t)\leqslant n(t)k_1(t). \tag{2.22}$$

Further, we may assume, without loss of generality, that $n(t)\geqslant m(t)$. Let $T\in I$ be any arbitrary number. Now from (2.20) we have for all $0\leqslant t\leqslant T$,

$$n(t)-m(t)=\Phi\Big(\int_0^t h(s)W(x(s))ds\Big)$$
$$\leqslant\Phi\Big(\int_0^t h(s)W(n(s)k_1(s))ds\Big)$$
$$\leqslant\Phi\Big(\int_0^t h(s)W(n(s))W(k_1(s))ds\Big)$$
$$\leqslant\Phi\Big(\int_0^t h(s)W(n(s)-m(s)+m(s))W(k_1(s))ds\Big)$$
$$\leqslant\Phi(h(s)W(h(s)-m(s))W(k_1(s))ds+$$
$$+\int_0^t h(s)W(m(s))W(k_1(s))ds).$$

Hence

$$n(t)-m(t)\leqslant\Phi\Big(\int_0^t h(s)W(n(s)-m(s))W(k_1(s))ds+$$
$$+\int_0^T h(s)W(m(s))W(k_1s))ds\Big) \tag{2.23}$$

using the submultiplicativity, subadditivity of W and monotonicity of Φ. Denote the expression in the parantheses in (2.23) by $v(t)$, then we have:

$$n(t)-m(t)\leqslant\Phi(v(t)), \tag{2.24}$$

where

$$v(t)=\int_0^t h(s)W(n(s)-m(s))W(k_1(s))ds+\int_0^T h(s)W(m(s))W(k_1(s))ds. \tag{2.25}$$

Hence

$$v'(t)\leqslant h(t)W(n(t)-m(t))W(k_1(t))\leqslant h(t)W(\Phi(v(t))W(k_1(t)),$$

which implies

$$\frac{v'(t)}{W(\Phi(v(t))}\leqslant h(t)W(k_1(t)), \qquad t\,[0,T], \tag{2.26}$$

Using (2.17), this further reduces to

$$\frac{d}{dt}G(v(t)\leqslant h(t))W(k_1(t)). \tag{2.27}$$

Integrating both sides from o to T and using (2.23) we obtain

$$n(T)-m(T)\leqslant\Phi\Big(\int_0^T h(s)W(n(s)-m(s))W(k_1(s))ds$$
$$+\int_0^T h(s)W(m(s))W(k_1(s))ds). \tag{2.28}$$

The conclusion of the theorem follows from (2.22) and (2.28).

THEOREM 4. *Let $x(t)$, $f(t)$, $g(t)$, $h(t)$ and $q(t)$ be real valued positive continuous function defined on I. The functions $W(t,u)$, $m(t)$, and $\Phi(t)$ are as defined before. Further suppose that the inequality*

$$x(t)\leqslant m(t)+h(t)\Phi\Big(\int_0^t q(s)W(s,x(s))ds+\int_0^t f(s)x(s)ds+\int_0^t g(s)\Big[m(s)$$
$$+h(s)\Phi\Big(\int_0^s q(\rho)W(\rho,x(\rho))d\rho\Big)\Big]^{1-p}\cdot x^p(s)ds, \tag{2.29}$$

holds. Then

$$x(t)\leqslant k_1(t)\,[m(t)+h(t)\Phi(r(t))], \tag{2.30}$$

for all $t\in I$, where k_1 is given by (1.3) and $r(t)$ is the maximal solution of

$$r(t)=q(t)W(t,k_1(t))\,[m(t)+h(t)\Phi(r(t))],\ r(0)=0 \tag{2.31}$$

existing on I.

PROOF. Define

$$n(t)=m(t)+h(t)\Phi\Big(\int_0^t q(s)W(s,x(s))ds\Big).$$

Then inequality (2.29) takes the form

$$x(t)\leqslant n(t)+\int_0^t f(s)x(s)ds+\int_0^t g(s).\,n^{1-p}(s).\ x^p(s)ds.$$

Since $n(t)$ is positive, monotonic, nondecreasing on I, and applying theorem A, we have

$$x(t)\leqslant n(t)k_1(t) \tag{2.32}$$

Let $v(t)=\int_0^t q(s)W(sx(s))ds,\qquad v(0)=0.$

Then

$$n(t)=m(t)+h(t)\Phi(v(t)).$$

Hence (2.32) takes the form

$$x(t)\leqslant k_1(t)\,[m(t)+h(t)\Phi(v(t))]. \tag{2.33}$$

Consequently it follows that

$$v'(t)=q(t)W(t,x(t))\leqslant q(t)W(t,k_1(t)\,[m(t)+h(t)\Phi(v(t))].$$

A suitable application of theorem (1.4.1) ([g], page 15) of lakshmikantham yields

$$v(t)\leqslant r(t), \tag{2.34}$$

where $r(t)$ is the maximal solution of (2.31) such that $r(0)=v(0)=0$. Hence from (2.33) we have

$$x(\mathrm{t})\leqslant k_1(t)\,[m(t)+h(t)\Phi(r(t)).$$

The desired bound (2.30) follows from (2.33) and (2.34). This completes the proof the theorem:

REMARK. We finally mention that the integral inequalities obtained in this paper allow us to study the stability, boundedness and asymptotic beviour of the solutions of a class of more general differential and integral equations [12]

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