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PREOPEN SETS AND RESOLVABLE SPACES

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Abstract: This paper presents solutions to some recent questions raised by Katetov about the collection of preopen sets in a topological space.

1. Introduction

In a recent paper Mashhour et al [5] presented and studied a series of questions raised by Katetov about the collection of preopen sets in a topological space. These questions are as follows:

K1: Find necessary and sufficient conditions under which every preopen set is open.

K2: Find conditions under which every dense-in-itself set is preopen.

K3: Find conditions under which the intersection of any two preopen sets is preopen.

K4: If $(X, \mathscr{T})$ is a topological space, let $\mathscr{T}^*$ denote the topology on X obtained by taking the collection of all preopen sets of $(X, \mathscr{T})$ as a subbase. Find conditions under which $\mathscr{T}^*$ is discrete.

Reilly and Vamanamurthy [7] continued the study of these questions and obtained a complete solution to K1 as well as partial solutions to the other questions. The purpose of the present paper is to provide complete solutions to K3 and K4, thus generalizing some results obtained in [7]. Furthermore, a solution to K2 is given.

2. Preliminaries

For any subset A of a topological space $(X, \mathscr{T})$ we denote the closure of A resp. the interior of A with respect to $\mathscr{T}$ by $\overline{A}$ resp. int A. The relative topology on a subset Y of $(X, \mathscr{T})$ is denoted by $\mathscr{T}|Y$. If A is a subset of $Y \subset X$, the closure of A resp. the interior of A with respect to $\mathscr{T}|Y$ is denoted by $\overline{A}^Y$ resp. $\mathrm{int}_Y A$.

DEFINITION 1. A subset S of $(X, \mathscr{T})$ is called

(i) an α-set, if $S \subset \mathrm{int}\ \overline{\mathrm{int}\ S}$,

(ii) a semi-open set, if $S \subset \overline{\mathrm{int}\ S}$,

(iii) a preopen set, if $S \subset \operatorname{int} \overline{S}$.

These notions were introduced by Njastad [6], Levine [4], and Mashhour et al [5] respectively. We denote the family of all α-sets in $(X, \mathscr{T})$ by $\mathscr{T}^{\alpha}$. Njastad [6] has shown that $\mathscr{T}^{\alpha}$ is a topology on X satisfying $\mathscr{T} \subset \mathscr{T}^{\alpha}$. Reilly and Vamanamurthy [8] proved that a subset of a space $(X, \mathscr{T})$ is an α-set if and only if it is semi-open and preopen. The families of all semi-open sets and of all preopen sets in $(X, \mathscr{T})$ are denoted by $\mathrm{SO}(X, \mathscr{T})$ and $\mathrm{PO}(X, \mathscr{T})$ respectively. In general one cannot expect the families $\mathrm{SO}(X, \mathscr{T})$ or $\mathrm{PO}(X, \mathscr{T})$ to be topologies. Njastad [6] has proved that $\mathrm{SO}(X, \mathscr{T})$ is a topology if and only if $(X, \mathscr{T})$ is extremally disconnected. Let us observe that both $\mathrm{SO}(X, \mathscr{T})$ and $\mathrm{PO}(X, \mathscr{T})$ are closed under forming arbitrary unions. Hence $\mathrm{PO}(X, \mathscr{T})$ is a topology if and only if the intersection of any two preopen sets is preopen.

The following simple characterization of preopen sets turns out to be very useful.

PROPOSITION 1. *For any subset S of a space $(X, \mathscr{T})$ the following are equivalent*:

(*i*) $S \in PO(X, \mathscr{T})$.

(*ii*) *There is a regular open set $G \subset X$ such that $S \subset G$ and $\overline{S} = \overline{G}$.*

(*iii*) *S is the intersection of a regular open set and a dense set.*

(*iv*) *S is the intersection of an open set and a dense set.*

PROOF. (i)⇒(ii): Let $S \in \mathrm{PO}(X, \mathscr{T})$, i.e. $S \subset \operatorname{int} \overline{S}$. Put $G := \operatorname{int} \overline{S}$. Then G is regular open with $S \subset G$ and $\overline{S} = \overline{G}$.

(ii)⇒(iii): Suppose (ii) holds. Put $D := S \cup (X - G)$. Then D is dense and $S = G \cap D$.

(iii)⇒(iv): Trivial.

(iv)⇒(i): Suppose $S = G \cap D$ with G open and D dense. Then $\overline{S} = \overline{G}$, hence $S \subset G \subset \overline{G} = \overline{S}$. Thus $S \subset \operatorname{int} \overline{S}$.

Obviously open sets and dense sets in $(X, \mathscr{T})$ are preopen. Proposition 1 immediately shows that the intersection of an open set and a preopen set is preopen again. Moreover, since $\mathrm{PO}(X, \mathscr{T}) = \mathrm{PO}(X, \mathscr{T}^{\alpha})$ (see Proposition 2 below) the intersection of any α-set and any preopen set is preopen.

The following result is due to Jankovic [3].

PROPOSITION 2. *Let* $(X, \mathcal{T})$ *be a topological space. Then*

(i) $(\mathcal{T}^{\alpha})^{\alpha}=\mathcal{T}^{\alpha}$

(ii) $SO(X, \mathcal{T})=SO(X, \mathcal{T}^{\alpha})$

(iii) $PO(X, \mathcal{T})=PO(X, \mathcal{T}^{\alpha})$.

Our next result concerning the existence of non-preopen sets can be easily shown. The proof is hence omitted.

PROPOSITION 3. *Let* $(X, \mathcal{T})$ *be a topological space,* $S\in PO(X, \mathcal{T})$ *and* x *be a point of* $\bar{S}$-int$\bar{S}$. *Then* $S\cup\{x\}\notin PO(X, \mathcal{T})$. *In particular, if* G *is regular open in* $(X, \mathcal{T})$ *and* x *a point of* $\bar{G}$-G *then* $G\cup\{x\}\notin PO(X, \mathcal{T})$.

3. Resolvable and irresolvable spaces

A space $(X, \mathcal{T})$ is called resolvable if there is a subset D of X such that D and X-D are both dense in $(X, \mathcal{T})$. A subset A of X is resolvable if the subspace $(A, \mathcal{T}|A)$ is resolvable. A space $(X, \mathcal{T})$ is called irresolvable if it is not resolvable. Resolvable spaces have been studied in a paper of Hewitt [2] in 1943. Clearly any resolvable space is dense-in-itself and each open subspace of a resolvable space is also resolvable. A space $(X, \mathcal{T})$ is said to be heriditarily irresolvable it does not contain a nonempty resolvable subset. Dense-in-itself heriditarily irresolvable spaces have been called SI-spaces by Hewitt [2].

Recall that $(X, \mathcal{T})$ is said to be submaximal if each dense subset is open. Reilly and Vamanamurthy ([7], Theorem 4) have shown that a space $(X, \mathcal{T})$ is submaximal if and only if PO$(X, \mathcal{T})=\mathcal{T}$, thus answering question K1. An MI-space is a dense-in-itself submaximal space. We have the following implications:

submaximal⇒heriditarily irresolvable⇒irresolvable

MI⇒submaximal

SI⇒heriditarily irresolvable.

THEOREM 1 ([2]). *Every topological space* $(X, \mathcal{T})$ *can be represented as a disjoint union* $X=F\cup G$ *where* F *is closed and resolvable and* G *is open and heriditarily irresolvable.* $(X, \mathcal{T})$ *is resolvable if and only if* $G=\phi$ *and* $(X, \mathcal{T})$ *is heriditarily irresolvable if and only if* $F=\phi$.

Note that it is not necessary to restrict ourselves to the class of dense-in-itself spaces as Hewitt [2] did. In addition, one easily checks that the representation of $(X, \mathcal{T})$ given by Theorem 1 is unique. It will henceforth be called the Hewitt-representation of $(X, \mathcal{T})$.

In 1969 El'kin [1] studied the class of dense-in-itself irresolvable spaces using ultrafilters. It is easily seen that Lemma 1 in [1] is also valid if the hypothesis "dense-in-itself" is dropped. Thus we have

THEOREM 2 ([1]). *For a space $(X, \mathcal{T})$ the following are equivalent:*

(i) $(X, \mathcal{T})$ contains an open, dense and heriditarily irresolvable subspace.

(ii) Every open ultrafilter on X is a base for an ultrafilter on X.

(iii) Every nonempty open set is irresolvable.

(iv) For each dense subset D of $(X, \mathcal{T})$ $\mathrm{int}D$ is dense.

(v) For every $A \subset X$, if $\mathrm{int}A = \phi$ then A is nowhere dense.

As a consequence of Theorem 2 we obtain a characterization of irresolvable spaces which seems to be new.

THEOREM 3. *A space $(X, \mathcal{T})$ is irresolvable if and only if $\{\mathrm{int}D \mid \overline{D} = X\}$ is a filterbase on X.*

PROOF. Suppose that $(X, \mathcal{T})$ is resolvable. Then there exist disjoint dense sets D, $E \subset X$ such that $X = D \cup E$. Hence $\mathrm{int}D = \phi$, a contradiction.

To prove the converse, let $(X, \mathcal{T})$ be irresolvable and let $X = F \cup G$ be the Hewitt-representation of $(X, \mathcal{T})$. Then G is nonempty. If D_1 and D_2 are both dense in $(X, \mathcal{T})$, then $D_1 \cap G$ and $D_2 \cap G$ are dense in G, hence nonempty. By Theorem 2 (iv) $G \cap \mathrm{int}(D_1 \cap D_2)$ is dense in G. Since F is resolvable, int $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \phi$ and $\mathrm{int}F \subset \overline{E}_1 \cap \overline{E}_2$. Let $D_3 := G \cap \mathrm{int}(D_1 \cap D_2) \cup E_1$. Then D_3 is clearly dense in $(X, \mathcal{T})$. Let us show that $\mathrm{int}D_3 \subset G \cap \mathrm{int}(D_1 \cap D_2)$. Let $x \in \mathrm{int}D_3$. If $x \in G \cap \mathrm{int}(D_1 \cap D_2)$ we are done. If $x \in E_1$ then there is an open neighbourhood U of x such that $U \subset F$ and $U \subset D_3$. Hence $U = U \cap F \subset E_1$, thus $U \cap (X - E_1) = \phi$. Since $E_2 \subset X - E_1$ we have $U \cap \overline{E}_2 = \phi$ and thus $U \cap \mathrm{int}F = \phi$, a contradiction. It follows that $\mathrm{int}D_3 \subset \mathrm{int}D_1 \cap \mathrm{int}D_2$ proving that $\{\mathrm{int}\ D \mid \overline{D} = X\}$ is a filterbase on X.

4. Results

In this section we present solutions to the questions K2, K3 and K4 and study some conesequences of the results obtained.

THEOREM 4. *For a space $(X, \mathcal{T})$ the following are equivalent:*

(i) $(X, \mathcal{T})$ *contains an open, dense and heriditarily irresolvable subspace.*
(ii) $PO(X, \mathcal{T}) \subset SO(X, \mathcal{T})$
(iii) $\mathcal{T}^{\alpha} = PO(X, \mathcal{T})$
(iv) $(X, \mathcal{T}^{\alpha})$ *is submaximal.*

PROOF.

(i)$\Rightarrow$(ii): Let $S \in \mathrm{PO}(X, \mathcal{T})$. By Proposition 1, $S = G \cap D$ where G is open and D is dense. By Theorem 2 (iv), $\mathrm{int}D$ is dense. Hence $\overline{\mathrm{int}\, S} = \mathrm{int}\ \overline{D \cap G} = \overline{G}$, consequently $S \subset G \subset \overline{G} = \overline{\mathrm{int}\ S}$. Thus $S \in \mathrm{SO}\ (X, \mathcal{T})$.

(ii)$\Rightarrow$(iii): Obvious, since $\mathcal{T}^{\alpha} = \mathrm{SO}(X, \mathcal{T}) \cap \mathrm{PO}(X, \mathcal{T})$ by a result of Reilly and Vamanamurthy [8].

(iii)$\Rightarrow$(iv): Follows from Proposition 2 (iii) and Theorem 4 in [7].

(iii)$\Rightarrow$(i): Let $D \subset X$ be dense. Then $D \in \mathrm{PO}(X, \mathcal{T}) = \mathcal{T}^{\alpha} \subset \mathrm{SO}(X, \mathcal{T})$. Consequently, $D \subset \overline{\mathrm{int}\ D}$, hence $\mathrm{int}D$ is dense. By Theorem 2 $(X, \mathcal{T})$ contains an open, dense and heriditarily irresolvable subspace.

COROLLARY 1. *If* $(X, \mathcal{T})$ *contains an open, dense and heriditarily irresolvable subspace then* $PO(X, \mathcal{T})$ *is a topology. In fact,* $PO(X, \mathcal{T}) = \mathcal{T}^{\alpha}$.

The following lemma will be useful in the sequel. Its proof is straightforward.

LEMMA 1. *Let* H *be an open subset of a space* $(X, \mathcal{T})$ *and let* $S \subset H$. *Then* $S \in PO(X, \mathcal{T})$ *if and only if* $S \in PO(H, \mathcal{T}|H)$.

As already pointed out, in proving that $\mathrm{PO}(X, \mathcal{T})$ is a topology it suffices to show that the intersection of any two preopen sets is preopen. Our next result generalizes this observation. Its proof follows from Proposition 1.

PROPOSITION 4. *For a space* $(X, \mathcal{T})$ *the collection* $PO(X, \mathcal{T})$ *is a topology if and only if the intersection of any two dense sets is preopen.*

THEOREM 5. *For a space* $(X, \mathcal{T})$ *let* $X = F \cup G$ *denote the Hewitt-representation of* $(X, \mathcal{T})$. *Then the following are equivalent:*
(i) $PO(X, \mathcal{T})$ *is a topology on* X.
(ii) $\overline{G}$ *is open and* $\{x\} \in PO(X, \mathcal{T})$ *for each* $x \in \mathrm{int}F$.

PROOF.

(i)$\Rightarrow$(ii): Since F is resolvable and closed $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \phi$ and $F = \overline{E}_1 = \overline{E}_2$. Let $y \in F$, say $y \in E_1$. Since $G \cup E_1$ and $G \cup E_2 \cup \{y\}$ are both dense, $(G \cup E_1) \cap (G \cup E_2 \cup \{y\}) \in G \cup \{y\} \in \mathrm{PO}(X, \mathcal{T})$. By Proposition 3 $\overline{G} - \mathrm{int}\overline{G} = \phi$, hence $\overline{G}$ is open.

Let $x \in \text{int}F$. Since $G \cup \{x\}$ and $\text{int}F$ are both preopen, $(G \cup \{x\}) \cap \text{int}\, F = \{x\} \in PO(X, \mathcal{T})$.

(ii)⇒(i): If S_1, $S_2 \in PO(X, \mathcal{T})$ then $S_1 \cap \overline{G}$ and $S_2 \cap \overline{G}$ are preopen, hence preopen in $\overline{G}$ by Lemma 1. Since G is an open, dense and heriditarily irresolvable subspace of $\overline{G}$, $(S_1 \cap \overline{G}) \cap (S_2 \cap \overline{G})$ is preopen in $\overline{G}$ by Theorem 4, hence preopen by Lemma 1. By assumption, $S_1 \cap S_2 \cap \text{int}F \in PO(X, \mathcal{T})$ and thus $S_1 \cap S_2 = (S_1 \cap S_2 \cap \overline{G}) \cup (S_1 \cap S_2 \cap \text{int}F) \in PO(X, \mathcal{T})$. Hence $PO(X, \mathcal{T})$ is a topology on X.

Let us consider some applications of Theorem 5.

COROLLARY 2. *For a resolvable space $(X, \mathcal{T})$ the following are equivalent:*
(i) $PO(X, \mathcal{T})$ is a topology.
(ii) Every subset of X is preopen.
(iii) Every open set is closed.

PROOF.
(i)⇒(ii): If $X = F \cup G$ denotes the Hewitt-representation of $(X, \mathcal{T})$ then by hypothesis $X = F = \text{int}F$. Hence $\{x\} \in PO(X, \mathcal{T})$ for every $x \in X$ by Theorem 5.
(ii)⇒(iii): This is Theorem 5 in [7].
(ii)⇒(i): Trivial.

Let $X = F \cup G$ denote the Hewitt-representation of $(X, \mathcal{T})$. Suppose there is some point $x \in \text{int}F$ such that $\{x\}$ is closed. By Theorem 5. $PO(X, \mathcal{T})$ fails to be a topology since F is dense-in-itself. In particular, we have

COROLLARY 3. *Let $(X, \mathcal{T})$ be a T_1-space. If $PO(X, \mathcal{T})$ is a topology then $PO(X, \mathcal{T}) = \mathcal{T}^\alpha$.*

PROOF. Since $(X, \mathcal{T})$ is a T_1-space $\text{int}F$ has to be void. Hence $\overline{G} = X$ and $PO(X, \mathcal{T}) = \mathcal{T}^\alpha$ by Theorem 4.

Note, however, that if $PO(X, \mathcal{T})$ is a topology then $PO(X, \mathcal{T}) \neq \mathcal{T}^\alpha$ in general. For an infinite indiscrete space $(X, \mathcal{T})$ we have $\mathcal{T}^\alpha = \mathcal{T}$ and $PO(X, \mathcal{T}) = 2^X$. Moreover, let X be an infinite set and pick a point $p \in X$. Then $\mathcal{T} = \{\phi, X, \{p\}, X - \{p\}\}$ is a topology on X. One easily checks that $(X, \mathcal{T})$ is irresolvable and $PO(X, \mathcal{T}) = 2^X$. However, for each nonempty proper subset S of $X - \{p\}$ we have $\text{int}S = \phi$, hence $S \notin \mathcal{T}^\alpha$.

COROLLARY 4. *Let $(X, \mathcal{T})$ be connected. If $PO(X, \mathcal{T})$ is a topology then $(X, \mathcal{T})$ is indiscrete or $PO(X, \mathcal{T}) = \mathcal{T}^\alpha$.*

PROOF. Let $X=F\cup G$ be the Hewitt-representation of $(X, \mathscr{T})$. Since $X=\overline{G}\cup \mathrm{int}F$ either $\overline{G}=\phi$ or $\mathrm{int}F=\phi$. If $\overline{G}=\phi$ then $(X, \mathscr{T})$ is resolvable. By Corollary 2, every open set is closed, hence $(X, \mathscr{T})$ has to be indiscrete. If $\mathrm{int}F=\phi$ then $\mathrm{PO}(X, \mathscr{T})=\mathscr{T}^{\alpha}$ by Theorem 4.

Recall that the topology on X having $\mathrm{PO}(X, \mathscr{T})$ as a subbase is denoted by $\mathscr{T}^{*}$.

LEMMA 2. *Let $X=F\cup G$ be the Hewitt-representation of $(X, \mathscr{T})$ and let x be a nonisolated point of G. Then $\{x\}\notin\mathscr{T}^{*}$.*

PROOF. Suppose that $\{x\}\in\mathscr{T}^{*}$. By Proposition 1 we have $\{x\}=U\cap D_1\cap D_2\cap\cdots D_k$, where U is open and each set D_i is dense in $(X, \mathscr{T})$. By Theorem 2, $\mathrm{int}\,(D_1\cap\cdots\cap D_k)\cap G$ is dense in G. Since $x\in V:=U\cap G$ is an open neighbourhood of x we have $V\cap\mathrm{int}(D_1\cap\cdots\cap D_k)\cap G\neq\phi$, . hence $\{x\}=V\cap\mathrm{int}(D_1\cap\cdots\cap D_k)$, a contradiction.

THEOREM 6. *Let $X=F\cup G$ denote the Hewitt-representation of $(X, \mathscr{T})$. Then the following are equivalent:*

(i) $\mathscr{T}^{*}$ is discrete.

(ii) F is open and $\{x\}$ is open for each $x\in G$.

PROOF

(i)⇒(ii): If $G=\phi$ we are done, so let us assume $G\neq\phi$. By Lemma 2, $\{x\}$ is open for each $x\in G$, hence $G\subset D$ for each dense set $D\subset X$. Let $x\in F$. Then $\{x\}=U\cap D_1\cap\cdots\cap D_k$ with U open and each D_i dense. Since $G\subset D_1\cap\cdots\cap D_k$, we have $G\cap U=\phi$, hence $U\subset F$. Thus $x\in\mathrm{int}F$, showing that F is open.

(ii)⇒(i): If $x\in G$ then clearly $\{x\}\in\mathscr{T}^{*}$. Let $x\in F$. Since F is resolvable, $F=E_1\cup E_2$ with $E_1\cap E_2=\phi$ and $F=\overline{E}_1=\overline{E}_2$. We may assume $x\in E_1$. Then $G\cup E_1$ and $G\cup E_2\cup\{x\}$ are both dense in $(X, \mathscr{T})$, hence $(G\cup E_1)\cap(G\cup E_2\cup\{x\})=G\cup\{x\}\in\mathscr{T}^{*}$. Consequently $(G\cup\{x\})\cap F=\{x\}\in\mathscr{T}^{*}$.

COROLLARY 5. *If a space $(X, \mathscr{T})$ is resolvable then $\mathscr{T}^{*}$ is discrete.*

COROLLARY 6. *Let $(X, \mathscr{T})$ be connected. Then $\mathscr{T}^{*}$ is discrete if and only if $(X, \mathscr{T})$ is resolvable.*

THEOREM 7. *For a space $(X, \mathscr{T})$ the following are equivalent:*

(i) $\mathscr{T}^{*}$ is discrete.

(ii) For each $x\in X$, either $\{x\}$ is open or there is a preopen set S such that

$x \in S$ *and* $\text{int}S = \phi$.

PROOF.

(i)⇒(ii): Let $X = F \cup G$ be the Hewitt-representation of $(X, \mathscr{T})$ and let $x \in X$. If $\{x\}$ is not open then $x \in F$ by Theorem 6. Since F is resolvable, $F = E_1 \cup E_2$ with $E_1 \cap E_2 = \phi$ and $\overline{E}_1 = \overline{E}_2 = F$. We may assume $x \in F_1$. Since F is open and $G \cup E_1$ is dense $S: = E_1 = (G \cup E_1) \cap F$ is preopen. Clearly $x \in S$ and $\text{int}S = \phi$.

(ii)⇒(i): If $\{x\}$ is open then $\{x\} \in \mathscr{T}^*$. Otherwise pick a preopen set S such that $x \in S$ and $\text{int}S = \phi$. Since $X - S$ is dense, $S \cap ((X - S) \cup \{x\}) = \{x\} \in \mathscr{T}^*$.

Finally let us consider question K2. Recall that a subset A of a space $(X, \mathscr{T})$ is called perfect if it is closed and dense-in-itself.

THEORFM 8. *For a space* $(X, \mathscr{T})$ *the following are equivalent*:

(*i*) *Every dense-in-itself subset is preopen.* .

(*ii*) *Every perfect subset is open.*

PROOF.

(i)⇒(ii): Let $A \subset X$ be perfect. By hypothesis, A is preopen, hence $A \subset \text{int}\overline{A} = \text{int}A$. Thus A is open.

(ii)⇒(i): Let $A \subset X$ be dense-in-itself. Then $\overline{A}$ is perfect. By hypothesis, $\overline{A}$ is open, hence $A \subset \overline{A} = \text{int}\overline{A}$. Thus A is preopen.

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