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F-CLOSED SPACES

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Abstract: The purpose of this paper is to introduce a topological space named an F-closed space. This space is properly contained between an S-closed space [17] and a quasi H-closed space [14], and between a nearly compact space [15] and a quasi H-closed space. We will investigate properties of F-closed spaces, and improve some results in [2], [7] and [17].

1. Introduction.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed. Let A be a subset of a space X. By $T(X)$, $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$ (T, $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ without confusions) we will denote, respectively, the topology on X, the closure of A and the interior of A in X. A subset A of a space X is said to be semiopen [N. Levine] if for some $U \in T(X)$, $U \subset A \subset \mathrm{cl}(U)$. By $\mathrm{SO}(X)$ and $s\mathrm{cl}(A)$ we will denote. respectively, the family of all semiopen sets and the semiclosure of A in X.

S.N. Maheshwari defined a subset $A \subset X$ to be feebly open [10] if for an $O \in T(X)$, $O \subset A \subset s\mathrm{cl}(O)$. The complement of a feebly open set is said to be feebly closed. The feeble closure of a set A ($f\mathrm{cl}(A)$) and the feeble interior of A ($f\mathrm{int}(A)$) in a space were known to be defined in manner analoguous to the standard concepts as well as the case of the closure and interior in a space. O. Njastad defined a set of a space X to be α-set [11] if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. It was shown in [4, Theorem 2.1.] that in every space, feebly open sets are the same sets as α-sets. We will denote by $\mathrm{FO}(X)$ the family of all feebly open sets in a space X.

A subset A is said to be regular open (resp. feebly regular open [4] and regular semiopen [2]) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = f\mathrm{int}(f\mathrm{cl}(A))$) and if for some regular open set U, $U \subset \mathrm{cl}(U) \subset A$). By $\mathrm{RO}(X)$, $\mathrm{FRO}(X)$ and $\mathrm{RSO}(X)$ we will denote, respectively, the families of all regular open sets, all feebly regular open sets and all regular semiopen sets of a space X.

In [14], authors defined a space X to be quasi H-closed (QHC) if every open

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cover of X has a finite proximate subcover. Similarly, T. Thompson defined a space X to be S-closed (SC) [17] if every semiopen cover of X has a finite proximate subcover. A space X is said to be nearly compact (NC) [16] if every regular open cover of X has a finite subcover. A space X is almost regular [15] if and only if, for each $x \in X$ and $V \in RO(X)$ containing x, there exists a $U \in RO(X)$ such that $x \in U \subset cl(U) \subset V$. A space X is said to be extremally disconnected (e.d.) if for each $O \in T(X)$, $cl(O) \in T(X)$.

2. Characterization

DEFINITION 2.1. A space X is said to be F-closed (FC) if every feebly open cover of X has a finite proximate subcover.

THEOREM 2.1. *A space X is FC if and only if every of X by feebly regular open sets has a finite proximate subcover.*

PROOF. One part follows from the fact that if $A = f\mathrm{int}(f\mathrm{cl}(A))$, then $A \in FO(X)$. Conversely, assume X is not FC. Then there exists a cover of X, $\mathscr{V} = \{V_\lambda | V_\lambda \in FO(X),\ \lambda \in D\}$ which has no finite proximate subcover. However, $\{f\mathrm{int}(f\mathrm{cl}(V_\lambda)) | \lambda \in D\}$ is a feebly regular open cover no finite proximate subcover because $V_\lambda \subset f\mathrm{int}(f\mathrm{cl}(V_\lambda)) \subset cl(V_\lambda)$ for each $\lambda \in D$. This contradicts.

It was shown in [4, Theorem 3.10] that $RO(X) \subset FRO(X) \subset RSO(X)$. Therefore, by [2, Theorem 1], [12, Theorem 1.3] and the above theorem, we obtain that FC spaces are properly contained between SC spaces and QHC spaces.

In order to make easy the treatment of FC spaces, we define the following.

DEFINITION 2.2. A subset A of a space X is said to be F-closed if it is FC as the subspace of X. A subset A of a space (X, T) is said to be F-closed relative to X if each family $FO(X)$ which covers A has a finite subfamily whose union is T-dense in A.

Every set FC relative to a space X may not, in general, be an FC subspace of X as shown by the next examples.

EXAMPLE 2.1. Let $X = [0, 1]$ be the subspace of the reals and $J = \left\{\frac{1}{n} | n \in Z^+\right\}$. Then J is FC relative to X, but not an FC space even though it is feebly closed in X.

EXAMPLE 2.2. Let R be the cocountable space of the reals and J the set in example 2.1. Then J is FC relative to R, but not an FC subspace, though it is closed in R.

LEMMA 2.1. *Lex X be a space. If $A \in FO(X)$ and $B \in FO(X)$, then $A \cap B \in FO(A)$.* [4, Corollary 2.2.]

THEOREM 2.2. *Let X be a space and $H \in FO(X)$. Then H is FC iff it is FC relative to X.*

PROOF. Let $\mathscr{V} = \{V_\lambda | V_\lambda \in \mathrm{FO}(X),\ \lambda \in D\}$ be a cover of an FC space H. Then, by Lemma 2.1, $\{H \cap V_\lambda | \lambda \in D\}$ is a feebly open cover of H because $H \in \mathrm{FO}(X)$. Since H is FC, we have a finite subfamily $\mathscr{V}_0$ of $\mathscr{V}$ whose union is dense in H with the relative topology. Conversely, let $\mathscr{V} = \{V_\lambda | V_\lambda \in \mathrm{FO}(H),\ \lambda \in D\}$ be a cover of H. Then, by Theorem 2.6 in [4], each $V_\lambda \in \mathrm{FO}(X)$ because $H \in \mathrm{FO}(X)$. Since H is FC relative to X, there exists a finite subfamily $\mathscr{V}_0$ of $\mathscr{V}$ such that $H = \bigcup_{i=1}^n \mathrm{cl}(V_{\lambda i})$. Therefore, we have $H = \bigcup_{i=1}^n \mathrm{cl}_H(V_{\lambda i})$.

COROLLARY 2.1. *Let X be a space, $A \in FO(X)$ and $B \subset X$. If A and B are FC relative to X, then* $\mathrm{cl}(A)$ *and* $\mathrm{int}(\mathrm{cl}(B))$ *are FC subspace of X.*

3. Filterbase characterizations of FC spaces.

DEFINITION 3.1. A filterbase $\mathscr{B} = \{B_\lambda\}$ on a space X is said to f-converge (resp. f-accumulate) to a point $p \in X$ if for each $V \in \mathrm{FO}(X)$ containing p (resp. and each $B_\lambda \in \mathscr{B}$), there exists a $B_\lambda \in \mathscr{B}$ such that $B_\lambda \subset \mathrm{cl}(V)$ (resp. $B_\lambda \cap \mathrm{cl}(V) \neq \phi$) [4].

THEOREM 3.1. *For a space X, the followings are equivalent.*

(a) X is FC

(b) Every ultra filterbase $\mathscr{B} = \{B_\lambda\}$ f-converges

(c) Every filterbase $\mathscr{B} = \{B_\lambda\}$ f-accumulates to a point p in X.

(d) For each family of feebly closed sets $\{F_\lambda\}$ such that $\bigcap_\lambda F_\lambda = \phi$, there exists a finite subfamily $\{B_{\lambda i} | i = 1, 2, \cdots, n\}$ such that $\bigcap_{i=1}^n \mathrm{int}(F_{\lambda i}) = \phi$.

PROOF. (a)$\Longrightarrow$(b): Let $\mathscr{B} = \{B_\lambda\}$ be an ultra filterbase on an FC space X. Assume that $\mathscr{B}$ does not f-converge to any point. Then for each $p \in X$, there exists a $V(p) \in \mathrm{FO}(X)$ containing p and a $B_\lambda \in \mathscr{B}$ such that $B_\lambda \cap \mathrm{cl}V(p)) = \phi$. Since $\{V(p) | V(p) \in \mathrm{FO}(X),\ p \in X\}$ is a cover of X, there exists a finite subfamily $\{V_{\lambda i}(p) | i = 1, 2, \cdots, n\}$ such that $X = \bigcup_{i=1}^n \mathrm{cl}(V_{\lambda i}(p))$. Since $\mathscr{B}$ is a filterbase, there exists a nonempty $B_0 \in \mathscr{B}$ such that $B_0 \subset \bigcap_{i=1}^n B_{\lambda i}(p)$. Thus $B_0 \cap \mathrm{cl}(V_{\lambda i}(p)) = \phi$, for each $i = 1, \cdots, n$. Thus $B_0 \cap [\bigcup_{i=1}^n \mathrm{cl}(V_{\lambda i}(p))] = B_0 \cap X = \phi$. This contradicts, for $B_0 \neq \phi$.

(b)$\Longrightarrow$(c)$\Longrightarrow$(d) were shown in [5, Theorem 2.7 and 2.9.].

(d)$\Longrightarrow$(a): Let $\{V_\lambda | V_\lambda \in FO(X),\ \lambda \in D\}$ be a cover of a space X. Then $\cap (X - V_\lambda) = \phi$. Since each $X - V_\lambda$ is feebly closed, by (d) there exists a finite snbfamily $\{V_{\lambda i} | i = 1, 2, \cdots, n\}$ such that $\cap_{i=1}^n \text{int}(X - V_{\lambda i}) = \cap_{i=1}^n (X - \text{cl}(V_{\lambda i})) = \phi$. Hence X is FC.

The following theorem shows that FC spaces are properly contained between NC space and QHC spaces.

THEOREM 3.2. *Every NC space is FC.*

PROOF. Let $\mathscr{V} = \{V_\lambda | V_\lambda \in FO(X),\ \lambda \in D\}$ be a cover of an NC space X. Then $X = \cup \text{int}(\text{cl}(\text{int}(V_\lambda)))$ because $\text{int}(V_\lambda) \subset V_\lambda \subset \text{int}(\text{cl}(\text{int}(V_\lambda)))$ for each $\lambda \in D$. Since X is NC and $\text{int}(V_\lambda) \in T(X)$, we have a finite subfamily $\{\text{int}(V_{\lambda i}) | \lambda_i \in D_0 \subset D\}$ such that $X = \cup_{i=1}^n\ \text{int}(\text{cl}(\text{int}(V_{\lambda i})))$. Thus $X = \cup_{i=1}^n \text{cl}(V_{\lambda i})$. Hence there exists a finite subfamily $\mathscr{V}_0 = \{V_{\lambda i} | \lambda i \in D_0 \subset D\}$ of $\mathscr{V}$ such that $X = \cup_{i=1}^n \text{cl}(V_{\lambda i})$. Thus X is FC.

COROLLARY 3.1. *Every almost regular QHC space is FC.*

PROOF. It is known that an almost regular space is QHC if and only if it is NC [16, Theorem 2.3].

It is known that compact spaces are NC. Thus, by Theorem 3.4, we obtain that compact spaces are FC. However, there exists an FC space which is not compact, as well as locally compact, as following examples show.

EXAMPLE 3.1. Let N be the set of natural numbers. For each $k \in N$, let $Y(k) = \{k + \frac{1}{n} | n = 2, 3, \cdots\}$ and $V(0,\ k) = \{0\} \cup \cup_{n=k}^\infty Y(n)$. Let $X = \cup_{k=1}^\infty Y(k) \cup N \cup \{0\}$ with the topology generated by the usual subspace topology of the reals on $\cup_{k=1}^\infty Y(k) \cup N$, and the set $\{V\ (0,\ k) | k \in N\}$. Then X is not compact because the infinte set N has no limit points. However, by Theorem 3.4, X is FC.

EXAMPLE 3.2. The one-point compactification $Q \cup \{\infty\}$ of Q in the reals R is FC, but not locally compact because Q is open in $Q \cup \{\infty\}$.

In Example 3.4, there exists an FC space which is not NC. However, we have the followings.

THEOREM 3.3. *Every e.d. FC space is NC.*

PROOF. It is easy to prove because for each $\lambda\in D$, $V_\lambda\in\mathrm{RO}(X)\subset\mathrm{FO}(X)$ and $\mathrm{cl}(V_\lambda)=\mathrm{int}(\mathrm{cl}(V_\lambda))=V_\lambda$.

THEOREM 3.4. *Every almost regular FC space is NC.*

PROOF. It is easy to prove and is thus omitted.

COROLLARY 3.2. *Every almost regular SC space is e.d. and NC* [7].

THEOREM 3.5. *Let X be an e.d. space. Then X is FC iff it is SC.*

PROOF. In an e.d. space X, $\mathrm{FO}(X)=SO(X)$ [4, Theorem 2.4].

From [2], [7], [17] and the aboves, we have the followings.

COROLLARY 3.3.

(*a*) *In an e.d. space, the following properties are equivalent:*

(1) *SC* (2) *FC* (3) *NC* (4) *QHC*

(*b*) *In a regular space, the following properties are equivalent:*

(1) *FC* (2) *NC* (3) *QHC* (4) *Compact*

EXAMPLE 3.3. βN is FC since it is SC [17, Corollary p. 337]. βQ is FC because it is compact Hausdorff, but not SC. βN-N is FC because it is closed in βN and hence βN-N is compact. However, βN-N is not SC [17].

EXAMPLE 3.4. Let X be the closed interval $[0,3]$ with the topology generated by the subspace topology along with the set $\{1\}$. Then X is not NC. However, X is FC because every feebly open cover of X has a finite proximate subcover and it is dense in X.

EXAMPLE 3.5. There exists a T_1 FC space which is not e.d., see [2, Example 3, p. 585].

4. Feebly continuous and feebly irresolute images, and the product of FC spaces.

We will denote by g: $X\to Y$ a function of a space X into a space Y. The following lemmas were shown in [3,] [4] and [5].

DEFINITION 4.1. A function g: $X\to Y$ is feebly continuous (resp. feebly irresolute [5]) [4] if for each $V\in T(X)$ (resp. $V\in\mathrm{FO}(Y)$), $g^{-1}(V)\in\mathrm{FO}(X)$.

LEMMA 4.1. *A function g: $X\to Y$ is feebly continuous (resp. feebly irresolute) iff for each $A\subset X$, $g(f\mathrm{cl}(A))\subset\mathrm{cl}(g(A))$ (resp. $g(f\mathrm{cl}(A))\subset f\mathrm{cl}(g(A))$).*

THEOREM 4.1. *Let a function g: $X \to Y$ is feebly continuous and onto. Then if X is FC, then Y is QHC.*

PROOF. *Let* $\mathscr{V}=\{V_\lambda | V_\lambda \in T(Y),\ \lambda \in D\}$ be a cover of Y. Then $\{g^{-1}(V_\lambda) | \lambda \in D\}$ is a feebly open cover of X. Since X is FC, we have a finite subfamily $\mathscr{V}_0$ of $\mathscr{V}$ such that $X=\cup_{i=1}^n \mathrm{cl}(g^{-1}(V_{\lambda i}))$. Since $\cup_{i=1}^n g^{-1}(V_{\lambda i})$ is dense in X, by Lemma 4.1 $Y=g(X)=g[f\mathrm{cl}(\cup_{i=1}^n g^{-1}(V_{\lambda i}))] \subset \mathrm{cl}[g(\cup_{i=1}^n (g^{-1}(V_{\lambda i})))] = \mathrm{cl}(\cup_{i=1}^n V_{\lambda i}) = \cup_{i=1}^n \mathrm{cl}(V_{\lambda i})$.

COROLLARY 4.1. *The continuous surjection of an FC space is QHC.*

COROLLARY 4.2. *The feebly continuous surjection of an FC space onto a regular space is compact.*

THEOREM 4.2. *Let g: $X \to Y$ be feebly irresolute and onto. Then if X is FC, then Y is FC.*

PROOF. The proof is almost similar to Theorem 4.1, using Lemma 4.1.

THEOREM 4.3. *Let X be FC. Then the continuous open image of X is FC.*

PROOF. Let $\mathscr{V}=\{V_\lambda | V_\lambda \in \mathrm{FO}(Y),\ \lambda \in D\}$ be a cover of $g(X)$, where g: $X \to Y$ is a continuous open function. Then $\mathscr{U}=\{g^{-1}(V_\lambda) | \lambda \in D\}$ is a cover of X. For each $\lambda \in D$, $g^{-1}(V_\lambda) \in \mathrm{FO}(X)$ because g is continuous and open [5, Theorem 2.6]. Hence $\mathscr{U}$ is a feebly open cover of an FC space X. Thus we have a finite subfamily $\{g^{-1}(V_{\lambda i}) | (\lambda i \in D_0 \subset D\}$ such that $X=\cup_{i=1}^n \mathrm{cl}(g^{-1}(V_{\lambda i}))$. By the continuity of g, we have a finite subfamily $\mathscr{V}_0$ of $\mathscr{V}$ such that $g(X) \subset \cup_{i=1}^n \mathrm{cl}(V_{\lambda i})$. Thus $g(X)$ is FC relative to Y. By Theorem 2.2, the proof completes.

LEMMA 4.2. *Let g: $X \to Y$ be a function of an e.d. space X to a Hausdorff space Y.*

PROOF. The proof follows from the fact if every filterbase on X f-accumulates to a point, then the feebly irresolute image of g is closed [4, Theorem 4.8 and Remark 4.1].

THEOREM 4.4. *The feebly irresolute image of any FC Hausdorff space X into any Hausdorff space Y is closed.*

PROOF. Since, by Theorem 3.3, any FC Hausdorff space is e.d., we can easily show that $g(X)=\mathrm{cl}[g(X)]$, utilizing Lemma 4.2 and Hausdorff property of Y, where g: $X \to Y$ is feebly irresolute.

THEOREM 4.5. *Let* $\{X_\lambda | \lambda \in D\}$ *be a family of spaces. If the product space* $X = \underset{\lambda \in D}{\pi} X_\lambda$ *is FC, then each* X_λ *is FC.*

PROOF. The natural projection is a continuous open surjection. Thus the natural projection is feebly irresolute [5, Theorem 2.5]. Therefore, each X_λ is FC, by Theorem 4.3.

The converse to Theorem 4.5 may not be true, in general, as the next example shows.

EXAMPLE 4.1. βN is FC, but $\beta N \times \beta N$ is not e.d. even though it is Hausdorff. Therefore, $\beta N \times \beta N$ is not FC, by Theorem 3.3.

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