Kyungpook Math. J.
Volume 27, Number 2
December, 1987

ON SOME SEPARATION PROPERTIES ON BITOPOLOGICAL SPACES

By Takayoshi Fukutake

1. Introduction

Many authors have studied several separation properties on a bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ which is the set X with two arbitrary topologies $\mathscr{T}_1$ and $\mathscr{T}_2$. In this paper we introduce some new separation properties on a bitopological space and investigate the relation among new separation properties and others. Notations will be explained when they are used for the first time.

2. Pairwise hausdorff spaces

W. Dunham [1 ; Def. 2.1] introduced and investigated the concept of weakly Hausdorff in topological spaces. In this section we introduce the concept of weakly Hausdorff in bitopological spaces and investigate the transfer of pairwise weakly Hausdorff conditions on a bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ to pairwise Hausdorff conditions on a bitopological space $(X;\ \mathscr{T}_1^*,\ \mathscr{T}_2^*)$. The following concept of pairwise Hausdorff conditions was introduced by J.D. Weston [10] as "consistent" and J.C. Kelly used the term "pairwise Hausdorff" in [4 ; Def. 2.5].

DEFINITION 2.1 (e.g. I.L. Reilly [8 ; Def. 4.1]). A bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is said to be pairwise Hausdorff if for each pair of distinct points x and y in X there are a $\mathscr{T}_1$-open set U and a $\mathscr{T}_2$-open set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

I.L. Reilly [8] showed that pairwise Hausdorff spaces can be characterized in terms of nets. A point x in X is called a $\mathscr{T}$-limit point of a net $\{x_\sigma;\ \sigma\in\Sigma\}$ on X if for every $\mathscr{T}$-open set G containing x there exists a $\sigma_0\in\Sigma$ such that $x_\sigma\in G$ for every $\sigma\geqq\sigma_0$ where $\mathscr{T}$ is a topology on X. The set of all $\mathscr{T}$-limit points of the net $\{x_\sigma;\ \sigma\in\Sigma\}$ is denoted by $\mathscr{T}$-lim x_σ.

Here we define the concept of pairwise weakly Hausdorff spaces. From now on $\mathscr{T}$-Cl(A) denotes the closure of A relative to a topology $\mathscr{T}$. Especially we denotes $\mathscr{T}$-Cl(x) insted of $\mathscr{T}$-Cl$(\{x\})$.

DEFINITION 2.2. $(X; \mathscr{T}_1, \mathscr{T}_2)$ is said to be pairwise weakly Hausdorff if $\mathscr{T}_1$-$\mathrm{Cl}(x)=\mathscr{T}_2$-$\mathrm{Cl}(y)$ whenever there is a net $\{x_\sigma;\ \sigma\in\Sigma\}$ on X with $\mathscr{T}_1$-lim $x_\sigma\ni x$ and $\mathscr{T}_2$-lim $x_\sigma\ni y$.

REMARK 2.3. The above definition coincides with the weakly Hausdorff condition in the usual sense introduced by W. Dunham [1; Def. 2.1], when we take $\mathscr{T}_1=\mathscr{T}_2$. That is, a topological space $(x; \mathscr{T})$ is said to be weakly Hausdorff if $\mathscr{T}$-$\mathrm{Cl}(x)=\mathscr{T}$-$\mathrm{Cl}(y)$ whenever there is a net $\{x_\sigma;\ \sigma\in\Sigma\}$ on X with $\mathscr{T}$-lim $x_\sigma\ni x, y$.

A pairwise weakly Hausdorff space, however, may fail to be pairwise Hausdorff as the following example shows.

EXAMPLE 2.4. Let $X=\{a, b, c\}$ and $\mathscr{T}_1=\mathscr{T}_2=\{\phi, \{a\}, \{b, c\}, X\}$. Then $(X; \mathscr{T}_1, \mathscr{T}_2)$ is pairwise weakly Hausdorff, but it is not pairwise Hausdorff.

To state our main theorem in this section, we recall the concepts of g-closed sets (N. Levine [5]) and generalized closure operator Cl^* (W. Dunham [3]).

DEFINITION 2.5. (N, Levine [5; Def. 2.1], W. Dunham [3; Def. 3.2, 3.6]). In a topological space $(X; \mathscr{T})$, a subset A is said to be g-closed if $\mathscr{T}$-$\mathrm{Cl}(A)\subset G$ whenever $A\subset G$ and $G\in\mathscr{T}$. Let $\mathscr{D}=\{A;\ A\subset X$ and A is g-closed$\}$. For any $E\subset X$, define $\mathrm{Cl}^*(E)=\bigcap\{A; E\subset A\in\mathscr{D}\}$ and $\mathscr{T}^*=\{E\subset X;\ \mathrm{Cl}^*(E^c)=E^c\}$ where E^c denotes the complement of E.

(2.6) (W. Dunham [3; Th. 3.5, 3.7 and 2.2]). (i) *Cl^* is a Kuratowski operator on X, i.e. $\mathscr{T}^*$ is the topology on X generated by Cl^* in the usual manner,*

(ii) $\mathscr{T}\subset\mathscr{T}^*$.

(iii) *For each $x\in X$, either $\{x\}$ is closed or $\{x\}^c$ is g-closed.*

We first prove the following result concerning the transfer of pairwise weakly Hausdorff conditions on $(X; \mathscr{T}_1, \mathscr{T}_2)$ to pairwise Hausdorff conditions on $(X; \mathscr{T}_1^*, \mathscr{T}_2^*)$.

THEOREM 2.7, *If $(X; \mathscr{T}_1, \mathscr{T}_2)$ is pairwise weakly Hausdorff, then $(X; \mathscr{T}_1^*, \mathscr{T}_2^*)$ is pairwise Hausdorff.*

PROOF. Let $\{x_\sigma;\ \sigma\in\Sigma\}$ be a net on X such that $\mathscr{T}_1^*$-lim $x_\sigma\ni x$ and $\mathscr{T}_2^*$-lim $x_\sigma\ni y$. By (2.6) (ii), we have $\mathscr{T}_1$-lim $x_\sigma\ni x$ and $\mathscr{T}_2$-lim $x_\sigma\ni y$. Then it follows by assumptions that $\mathscr{T}_1$-$\mathrm{Cl}(x)=\mathscr{T}_2$-$\mathrm{Cl}(y)$.

Case 1, Suppose $\{x\}$ is $\mathscr{T}_1$-closed. Since $\{x\}=\mathscr{T}_1\text{-Cl}(x)=\mathscr{T}_2\text{-Cl}(y)\supset\{y\}$, we have $x=y$.

Case 2. Suppose $\{y\}$ is $\mathscr{T}_2$-closed, We assert that $x=y$ in the same manner as above.

Case 3. Suppose $\{x\}$ is not $\mathscr{T}_1$-closed and $\{y\}$ is not $\mathscr{T}_2$-closed. By using (2.6) (iii) we have $\{x\}\in\mathscr{T}_1^*$ and $\{y\}\in\mathscr{T}_2^*$. Since $\mathscr{T}_1^*\text{-lim } x_\sigma\ni x$, for $\{x\}\in\mathscr{T}_1^*$ there exists a $\sigma_1\in\Sigma$ such that $x_\sigma\in\{x\}$ (i.e. $x_\sigma=x$) for every $\sigma\geqq\sigma_1$. Similarly there exists a $\sigma_2\in\Sigma$ such that $x_\sigma\in\{y\}$ (i.e. $x_\sigma=y$) for every $\sigma\geqq\sigma_2$. From the fact that Σ is a directed set it follows that $\{x\}\cap\{y\}\neq\phi$ i.e. $x=y$. Hence $\mathscr{T}_1^*\text{-lim } x_\sigma\ni x$ and $\mathscr{T}_2^*\text{-lim } x_\sigma\ni y$ imply that $x=y$ in any cases. Therefore $(X;\mathscr{T}_1^*,\mathscr{T}_2^*)$ is pairwise Hausdorff by [8; Cor. 4.5].

REMARK 2.8. By setting $\mathscr{T}_1=\mathscr{T}_2$ in Th. 2.7, we have the theorem due to W. Dunham [3; Th. 4.10], namely, if $(X;\mathscr{T})$ is weakly Hausdorff then $(X;\mathscr{T}^*)$ is Hausdorff.

DEFINITION 2.9. $(X;\mathscr{T}_1,\mathscr{T}_2)$ is said to be bi-weakly Hausdorff if $\mathscr{T}_2\text{-Cl}(x)=\mathscr{T}_1\text{-Cl}(y)$ whenever there is a net $\{x_\sigma;\sigma\in\Sigma\}$ on X with $\mathscr{T}_1\text{-lim } x_\sigma\ni x$ and $\mathscr{T}_2\text{-lim } x_\sigma\ni y$.

PROPOSITION 2.10. *Let $(X;\mathscr{T}_1,\mathscr{T}_2)$ be a bi-weakly Hausdorff space which satisfies the following condition: a subset $\{x\}$ of X is $\mathscr{T}_1^*$-open if and only if the set $\{x\}$ is $\mathscr{T}_2^*$-open. Then $(X;\mathscr{T}_1^*,\mathscr{T}_2^*)$ is pairwise Hausdorff.*

PROOF. The proof is similar to one of Th. 2.7.

In the rest of this section we consider several properties concerning pairwise Hausdorff spaces.

THEOREM 2.11. *If $(X;\mathscr{T}_1,\mathscr{T}_2)$ is pairwise Hausdorff, then $\mathscr{T}_1\text{-lim } x_\sigma=\mathscr{T}_2\text{-lim } x_\sigma$ holds for each net $\{x_\sigma;\sigma\in\Sigma\}$ on X such that $\mathscr{T}_i\text{-lim } x_\sigma\neq\phi$ $(i=1,2)$.*

PROOF. Let $\mathscr{T}_1\text{-lim } x_\sigma\ni x$. Then there exists a $y\in\mathscr{T}_2\text{-lim } x_\sigma$ because $\mathscr{T}_2\text{-lim } x_\sigma\neq\phi$. By [8; Cor. 4.5], we have $x=y$ and so $\mathscr{T}_1\text{-lim } x_\sigma\subset\mathscr{T}_2\text{-lim } x_\sigma$. Similarly we have $\mathscr{T}_2\text{-lim } x_\sigma\subset\mathscr{T}_1\text{-lim } x_\sigma$. Thus $\mathscr{T}_1\text{-lim } x_\sigma=\mathscr{T}_2\text{-lim } x_\sigma$.

REMARK 2.12. The converse of Th. 2.11. is not always true by Example 2.4. Even if $(X;\mathscr{T}_1,\mathscr{T}_2)$ is pairwise Hausdorff, a net $\{x_\sigma;\sigma\in\Sigma\}$ on X has not always the same limit set $\mathscr{T}_1\text{-lim } x_\sigma=\mathscr{T}_2\text{-lim } x_\sigma$ as the following example shows.

EXAMPLE 2.13. Let X be the set of all real numbers R and let $\mathscr{T}_1$ be the discrete topology and $\mathscr{T}_2$ be the usual topology. Then $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is pairwise Hausdorff. However $\mathscr{T}_1$-lim $1/n=\phi$ and $\mathscr{T}_2$-lim $1/n=\{0\}$ for a net $\{1/n;\ n\in N\}$ on $X=R$, where N is the set of all natural numbers.

PROPOSITION 2.14. *Let* $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ *be a bitopological space such that* $\mathscr{T}_1\subset\mathscr{T}_2$ (*e.g.* $\mathscr{T}_2=\mathscr{T}_1^*$).

(*i*) *If* $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ *is pairwise Hausdorff, then* $(X;\ \mathscr{T}_2)$ *is Hausdorff.*

(*ii*) *If* $(X;\ \mathscr{T}_1)$ *is Hausdorff, then* $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ *is pairwise Hausdorff.*

PROOF. (i) Let x and y be in X such that $x\neq y$. By the hypothesis there exist $U\in\mathscr{T}_1$ and $V\in\mathscr{T}_2$ such that $x\in U$, $y\in V$ and $U\cap V=\phi$. Since $\mathscr{T}_1\subset\mathscr{T}_2$, we have $(X;\ \mathscr{T}_2)$ is Hausdorff.

(ii) is proved by the assumption.

The converse of Prop. 2.14 (i) (resp. (ii)) is not true in general by the following Example 2.15 (resp. 2.16).

EXAMPLE 2.15. Let $X=R$ and let $\mathscr{T}_1$ be the indiscrete topology and $\mathscr{T}_2$ be the discrete topology. Then $(X;\ \mathscr{T}_2)$ is Hausdorff but $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is not pairwise Hausdorff.

EXAMPLE 2.16. Let $X=R$ and let $\mathscr{T}_1=\{\phi, R\}\cup\{A;\ A\subset R$ and A^c is finite$\}$ and $\mathscr{T}_2$ be the discrete topology. Then $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is pairwise Hausdorff but $(X;\ \mathscr{T}_1)$ is not Hausdorff.

3. Bitopological spaces $(X;\ \mathscr{T},\ \mathscr{T}^\alpha)$

In this section we give a property of pairwise Hausdorff bitopological space $(X;\ \mathscr{T},\ \mathscr{T}^\alpha)$.

O. Njåstad [6] introduced a weak form of open sets called α-sets.

DEFINITION 3.1. (O. Njåstad [6]). Let A be a subset of a topological space $(X;\ \mathscr{T})$. The subset A is said to be α-open if $A\subset\mathscr{T}$-Int $(\mathscr{T}$-Cl $(\mathscr{T}$-Int$(A)))$ where $\mathscr{T}$-Int(A) denotes interior of A relative to $\mathscr{T}$. We denote the family of all α-open sets by $\mathscr{T}^\alpha$.

It is known that $\mathscr{T}^\alpha$ is a topology on X and $\mathscr{T}\subset\mathscr{T}^\alpha$ ([6]). Then $(X;\ \mathscr{T},\ \mathscr{T}^\alpha)$ is a bitopological space.

I.L. Reilly and M.K. Vamanamurthy [9] proved the following theorem. Recently, T. Noiri [7; Remark 4.8] gave a simple proof of necessity of the condition.

(3.2) (I.L. Reilly and M.K. Vamanamurthy [9 : Th. 2]). *$(X;\ \mathscr{T})$ is Hausdorff if and only if $(X;\mathscr{T}^{\alpha})$ is Hausdorff.*

DEFINITION 3.3. A bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is said to be bi-Hausdorff if both $(X;\ \mathscr{T}_1)$ and $(X;\ \mathscr{T}_2)$ are Hausdorff.

(3.4) *In a bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$, the two concepts of pairwise Hausdorff and bi-Hausdorff are independent.*

The following example is bi-Hausdorff, but it is not pairwise Hausdorff. Let X be an infinite set and let $\mathscr{T}_1=\{A\subset X;\ x_0\notin A$ or A^c is finite$\}$ and $\mathscr{T}_2=\{A\subset X;\ y_0\notin A$ or A^c is finite$\}$, where x_0 and y_0 are fixed distinct points of X. Example 2.16 is not bi-Hausdorff but pairwise Hausdorff.

To state Prop. 3.7 below, we recall the concepts of $T_{1/2}$-spaces (N. Levine [5]) and pairwise semi-Hausdorff spaces (I.L. Reilly [8]).

DEFINITION 3.5 (N. Levine [5 ; Def. 5.1]). A topological space $(X;\ \mathscr{T})$ is said to be a $T_{1/2}$-space if every g-closed set is closed.

DEFINITION 3.6 (I.L. Reilly [8 ; Def. 4.6]). A bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is said to be a pairwise semi-Hausdorff space if for each pair of distinct points x and y in X there exist $\mathscr{T}_1$-open set U and a $\mathscr{T}_2$-open set V disjoint from U such that either $x\in U,\ y\in V$ or $x\in V,\ y\in U$.

PROPOSITION 3.7. (i) *$(X;\ \mathscr{T},\ \mathscr{T}^{\alpha})$ is pairwise Hausdorff if and only if $(X;\ \mathscr{T},\ \mathscr{T}^{\alpha})$ is bi-Hausdorff.*

(ii) Let $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ be a bitopological space which satisfies that $(X;\ \mathscr{T}_1)$ is a $T_{1/2}$-space and that $\{x\}$ is $\mathscr{T}_2$-open whenever $\{x\}$ is $\mathscr{T}_1$-open or $\mathscr{T}_1$-closed. Then $(X;\ \mathscr{T}_1^{\alpha},\ \mathscr{T}_2^{\alpha})$ is pairwise semi-Hausdorff if and only if $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is pairwise semi-Hausdorff.

PROOF. (i) Sufficiency of the condition follows from the fact that $\mathscr{T}\subset\mathscr{T}^{\alpha}$. To prove necessity, let $(X;\ \mathscr{T},\ \mathscr{T}^{\alpha})$ be a pairwise Hausdorff bitopological space. It follows from Prop. 2.14 (i) that $(X;\ \mathscr{T}^{\alpha})$ is Hausdorff. By (3.2), $(X;\ \mathscr{T})$ is Hausdorff. Thus $(X;\ \mathscr{T},\ \mathscr{T}^{\alpha})$ is bi-Hausdorff.

(ii) Sufficiency follows from the fact that $\mathscr{T}_i\subset\mathscr{T}_i^{\alpha}$ for $i=1,2$. To prove necessity let x and y be in X such that $x\neq y$. Since $(X;\ \mathscr{T}_1^{\alpha},\ \mathscr{T}_2^{\alpha})$ is pairwise semi-Hausdorff, there exist a $\mathscr{T}_1^{\alpha}$-open set U and a $\mathscr{T}_2^{\alpha}$-open set V disjoint from U such that either $x\in U,\ y\in V$ or $x\in V,\ y\in U$.

Case 1. Suppose $x\in U$ and $y\in V$. It is known that either $\{x\}$ is $\mathscr{T}_1$-open or $\mathscr{T}_1$-closed by W. Dunham [2; Th. 2.5].

(1) Let $\{x\}$ be $\mathscr{T}_1$-open. We have the $\mathscr{T}_1$-open set $G=\{x\}$ and the $\mathscr{T}_2$-open set $H=\mathscr{T}_2\text{-Int}(\mathscr{T}_2\text{-cl}(\mathscr{T}_2\text{-Int}(V)))$ such that $x\in G$, $y\in H$ and $G\cap H=\phi$.

(2) Let $\{x\}$ be $\mathscr{T}_1$-closed. We have the $\mathscr{T}_1$-open set $\{x\}^C$ and the $\mathscr{T}_2$-open set $\{x\}$ which satisfy $x\in\{x\}$, $y\in\{x\}^C$ and $\{x\}\cap\{x\}^C=\phi$.

Case 2. Suppose $x\in V$ and $y\in U$. We have two open sets similarly as case 1. Then we conclude that $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is pairwise semi-Hausdorff.

4. $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regular spaces

In this section we investigate some relations between nearly regular spaces and nearly Hausdorff spaces and study the transfer of $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regularity to $(\mathscr{T}_i^*;\ \mathscr{T}_j^*,\ \mathscr{T}_k^*)$-regularity. We first define the concepts of $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regular spaces which contain the definition due to J.C. Kelly [4].

DEFINITION 4.1. A bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is said to be $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regular if for each point x in X and each $\mathscr{T}_i$-closed subset F such that $x\notin F$, there exist a $\mathscr{T}_j$-open set V and a $\mathscr{T}_k$-open set U such that $F\subset V$, $x\in U$ and $U\cap V=\phi$ where i, j and $k\in\{1,2\}$.

J.C. Kelly [4] says that $\mathscr{T}_i$ is regular with respect to $\mathscr{T}_j$ if $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_i)$-regular.

The following proposition concerning $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regularity contains Prop. 5.2 in [8] as a special case.

PROPOSITION 4.2. (c.f. I.L. Reilly [8; Prop. 5.2]. *Let i, j and k be fixed integers of* $\{1,\ 2\}$.

(*i*) *The following* (1)—(3) *are equivalent to each other.*

(1) $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ *is* $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-*regular.*

(2) *For each point x in X and $\mathscr{T}_i$-open set G such that $x\in G$ there exists a $\mathscr{T}_k$-open set H such that* $x\in H\subset\mathscr{T}_j\text{-cl}(H)\subset G$.

(3) *For each point x in X and $\mathscr{T}_i$-closed set K such that $x\notin K$ there exists a $\mathscr{T}_k$-open set M such that $x\in M$ and* $(\mathscr{T}_j\text{-cl}(M))\cap K=\phi$.

(*ii*) *Suppose $\mathscr{T}_k=\mathscr{T}_j$ in* (*i*) *Then each of above* (1)—(3) *is equivalent to the following* (4).

(4) *For each point x in X and $\mathscr{T}_i$-closed set F such that $x\notin F$ there exists a $\mathscr{T}_j$-open set V such that $F\subset V$ and* $x\notin\mathscr{T}_j\text{-cl}(V)$.

PROOF. (i) is equal to Prop. 5.2 of [8] when $\mathscr{T}_k=\mathscr{T}_i$. The proofs for other cases are similar to one of the proposition.

(ii) is proved by assumption and definitions.

We note that $(\mathscr{T}_1;\ \mathscr{T}_2,\ \mathscr{T}_2)$-regular spaces are not always $(\mathscr{T}_2;\ \mathscr{T}_1,\ \mathscr{T}_1)$-regular.

EXAMPLE 4.3. Let $X=R$ and let $\mathscr{T}_1=\{\phi,\ R\}\cup\{A;\ A\subset R$ and A^C is finite$\}$ and $\mathscr{T}_2$ be usual topology on R. Then $(R;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is not $(\mathscr{T}_2;\ \mathscr{T}_1,\ \mathscr{T}_1)$-regular but $(\mathscr{T}_1;\ \mathscr{T}_2,\ \mathscr{T}_2)$-regular.

Now we give a definition of another nearly regular space.

DEFINITION 4.4. A bitopological space $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is said to be $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-strongly regular if for each point x in X and $\mathscr{T}_i$-closed set F such that $x\notin F$ there exist a $\mathscr{T}_j$-open set V and a $\mathscr{T}_k$-open set U such that $x\in U$, $F\subset V$ and $U\cap(\mathscr{T}_j\text{-cl}(V))=\phi$ where $i,\ j$ and $k\in\{1,\ 2\}$.

REMARK 4.5. The bitopological space $(R;\ \mathscr{T}_1,\ \mathscr{T}_2)$ of Example 2.13 is $(\mathscr{T}_2;\ \mathscr{T}_1,\ \mathscr{T}_1)$-strongly, but it is not $(\mathscr{T}_1;\ \mathscr{T}_2,\ \mathscr{T}_2)$-strongly regular.

We have the following proposition concerning the nearly regularities.

PROPOSITION 4.6. *Let $i,\ j$ and $k\in\{1,\ 2\}$.*

(i) *If $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_j)$-regular, then $\mathscr{T}_i\subset\mathscr{T}_j$.*

(ii) *If $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_i)$-regular and $\mathscr{T}_i\subset\mathscr{T}_j$ then it is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_j)$-regular.*

(iii) *If $(X;\ \mathscr{T}_i)$ is a T_3-space and $\mathscr{T}_i\subset\mathscr{T}_j$ then $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_j)$-regular.*

(iv) *If $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-strongly regular then it is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regular. The converse is true whenever $\mathscr{T}_k=\mathscr{T}_j$.*

(v) *Jf $(X;\ \mathscr{T}_1,\ \mathscr{T}_2)$ is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-strongly regular then it is $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_j)$-regular.*

PROOF. (i) Let G be $\mathscr{T}_i$-open. For each $x\in G$ there exists a $\mathscr{T}_j$-open set H_x such that $x\in H_x\subset G$ by Prop. 4.2. This implies $G=\cup\{H_x;\ x\in G\}$ and therefore G is $\mathscr{T}_j$-open.

(ii) (iv) and (v) follow from definitions.

(iii) is proved by using (ii).

REMARK 4.7. The converse of Prop. 4.6(iv) and (v) are not true generally as the following examples show. Let $X=\{a, b, c\}$ and let $\mathcal{T}_1=\{\phi, \{a\}, X\}$ and $\mathcal{T}_2=\{\phi, \{b, c\}, X\}$. Then the bitopological space $(X; \mathcal{T}_1, \mathcal{T}_2)$ is not $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_1)$-strongly regular but $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_1)$-regular. The bitopological space $(R; \mathcal{T}_1, \mathcal{T}_2)$ in Example 4.3 is $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_2)$-regular, but it is not $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_1)$-strongly regular. The condition $\mathcal{T}_i \subset \mathcal{T}_j$ in Prop. 4.6(ii) can not be removed as the following example shows.

EXAMPLE 4.8. Let $X=R$ and let $\mathcal{T}_1=\{\phi, R\} \cup \{G; G\subset R$ and $(-\infty, x]\subset G$ for any $x\in G\}$ and $\mathcal{T}_2=\{\phi, R\} \cup \{G: G\subset R$ and $[x, +\infty)\subset G$ for any $x\in G\}$. Then $(R; \mathcal{T}_1, \mathcal{T}_2)$ is a bitopological space which is $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_1)$-regular but not $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_2)$-regular by Prop. 4.6(i).

(4.9) *$(\mathcal{T}_i; \mathcal{T}_j, \mathcal{T}_k)$-regularity is independent of pairwise Hausdorff conditions and pairwise semi-Hausdorff conditions.*

(4.10) *$(\mathcal{T}_i; \mathcal{T}_j, \mathcal{T}_k)$-strongly regularity is independent of pairwise Hausdorff conditions and pairwise semi-Hausdorff conditions.*

In fact, following examples show (4.9) and (4.10). Let $X=\{a, b, c\}$ and $\mathcal{T}_1=\{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{T}_2=\{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then $(X; \mathcal{T}_1, \mathcal{T}_2)$ is not pairwise semi-Hausdorff but $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_1)$-strongly regular. The bitopological space $(X; \mathcal{T}_1, \mathcal{T}_2)$ in Example 2.13 is not $(\mathcal{T}_1: \mathcal{T}_2, \mathcal{T}_2)$-regular but pairwise Hausdorff. The bitopological space $(X; \mathcal{T}_1, \mathcal{T}_2)$ in Example 4.3 is not pairwise semi-Hausdorff but $(\mathcal{T}_1; \mathcal{T}_2, \mathcal{T}_2)$-strongly regular.

THEOREM 4.11. *Let $i, j\in\{1, 2\}$ be fixed integers.*

(i) If a bitopological space $(X; \mathcal{T}_1, \mathcal{T}_2)$ is $(\mathcal{T}_i; \mathcal{T}_j, \mathcal{T}_j)$-regular and the topological space $(X; \mathcal{T}_i)$ is a $T_{1/2}$-space, then $(X; \mathcal{T}_j)$ is a Hausdorff space.

(ii) Especially if we can take $\mathcal{T}_j=\mathcal{T}_i^\alpha$ in (i), then $(X; \mathcal{T}_i, \mathcal{T}_i^\alpha)$ is pairwise Hausdorff.

(iii) If $(X; \mathcal{T}_1, \mathcal{T}_2)$ is $(\mathcal{T}_i; \mathcal{T}_j, \mathcal{T}_i)$-regular and $(X; \mathcal{T}_i)$ is a T_1-space (resp. $T_{1/2}$-space) then $(X: \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Hausdorff (resp. pairwise semi-Hausdorff).

PROOF. (i) Suppose $(X, \mathcal{T}_j)$ is not Hausdorff. Then there exists a net $\{x_\sigma: \sigma\in\Sigma\}$ on X such that $\mathcal{T}_j$-lim x_σ contains distinct points x and y. It is known that either $\{y\}$ is $\mathscr{T}_i$-closed or $\mathscr{T}_i$-open by W. Dunham [2; Th. 2.5].

Case 1. Suppose $\{y\}$ is $\mathscr{T}_i$-closed. Since $x \notin \{y\}$ there exists a $\mathscr{T}_j$-open V such that $\{y\} \subset V$ and $x \notin (\mathscr{T}_j\text{-Cl}(v))$ by Prop. 4.2(ii). It follows from the fact $\mathscr{T}_j\text{-lim}\, x_\sigma \ni y$ that there exists $\sigma_1 \in \Sigma$ such that $x_\sigma \in V$ whenever $\sigma \geqq \sigma_1$, $\sigma \in \Sigma$. Similarly there exists $\sigma_2 \in \Sigma$ such that $x_\sigma \in (\mathscr{T}_j\text{-Cl}(V))^C$ whenever $\sigma \geqq \sigma_2$, $\sigma \in \Sigma$. Take $\sigma_0 \in \Sigma$ such that $\sigma_1 \leqq \sigma_0$ and $\sigma_2 \leqq \sigma_0$ then $x_{\sigma_0} \in V \cap (\mathscr{T}_j\text{-Cl}(V))^C$ and this contradicts to the fact that $V \cap (\mathscr{T}_j\text{-Cl}(V))^C = \phi$.

Case 2. Suppose $\{y\}$ is $\mathscr{T}_i$-open. Put $F = \{y\}^C$ then F is $\mathscr{T}_i$-closed and $y \notin F$. W obtain a contradiction similarly as Case 1.

Then it comes to the conclusion that $(X; \mathscr{T}_j)$ is Hausdorff.

(ii) The proof follows from (i), (3.2) and Prop. 3.7(i).

(iii) Let $(X; \mathscr{T}_1, \mathscr{T}_2)$ be $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_i)$-regular and $(X; \mathscr{T}_i)$ be a T_1-space. Then for each distinct pair x and y in X $\{y\}$ is $\mathscr{T}_i$-closed and $x \notin \{y\}$. Therefore there exist a $\mathscr{T}_i$-open set U and a $\mathscr{T}_j$-open set V such that $x \in U$, $\{y\} \subset V$ and $U \cap V = \phi$. Then $(X; \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorff. Similarly we have the proof of the case which $(X; \mathscr{T}_1, \mathscr{T}_2)$ is $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_i)$-regular and $(X; \mathscr{T}_i)$ is a $T_{1/2}$-space.

THEOREM 4.12. *Let i, j and $k \in \{1, 2\}$ be fixed integers.*

(i) If $(X; \mathscr{T}_1, \mathscr{T}_2)$ is $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_j)$-regular and $(\mathscr{T}_j; \mathscr{T}_i, \mathscr{T}_j)$-regular then $(X; \mathscr{T}_j^*)$ is a Hausdorff space.

(ii) If $(X; \mathscr{T}_1, \mathscr{T}_2)$ is $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_k)$-regular and $(X; \mathscr{T}_i)$ is $T_{1/2}$-space, then $(X; \mathscr{T}_1^*, \mathscr{T}_2^*)$ is $(\mathscr{T}_i^*; \mathscr{T}_j^*, \mathscr{T}_k^*)$-regular.

(iii) If $(X; \mathscr{T}_1, \mathscr{T}_2)$ is $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_k)$-regular and satisfies the following conditions:

(4.13) *a subset $\{x\}$ of X is $\mathscr{T}_h^*$-open for $h \in \{1,2\}$ whenever $\{x\}$ is $\mathscr{T}_i^*$-open, then $(X; \mathscr{T}_1^*, \mathscr{T}_2^*)$ is $(\mathscr{T}_i^*; \mathscr{T}_j^*, \mathscr{T}_k^*)$-regular.*

PROOF. (i) Let x and y be two distinct points of X.

Case 1. Suppose $\{x\}$ is $\mathscr{T}_j$-closed. Since $(X; \mathscr{T}_1, \mathscr{T}_2)$ is $(\mathscr{T}_j; \mathscr{T}_i, \mathscr{T}_j)$-regular, there exist a $\mathscr{T}_j$-open set U and a $\mathscr{T}_i$-open set V such that $U \cap V = \phi$, $y \in U$ and $\{x\} \subset V$. It follows from $(\mathscr{T}_i; \mathscr{T}_j, \mathscr{T}_j)$-regularity, Prop. 4.6(i) and (2.6) (ii) that U and V are $\mathscr{T}_j^*$-open.

Case 2. Suppose $\{y\}$ is $\mathscr{T}_j$-closed. We have $\mathscr{T}_j^*$-open sets U and V such that $U \cap V = \phi$, $x \in U$ and $y \in V$ as similarly as above.

Case 3. Suppose $\{x\}$ is not $\mathscr{T}_j$-closed and $\{y\}$ is not $\mathscr{T}_j$-closed. It follows from (2.6) (iii) that $\{x\}$ and $\{y\}$ are $\mathscr{T}_j^*$-open.

From above discussion we conclude that $(X;\ \mathscr{T}_j^*)$ is Hausdorff.

(ii) Since $(X;\ \mathscr{T}_i)$ is a $T_{1/2}$-space we have $\mathscr{T}_i=\mathscr{T}_i^*$ (W. Dunham [3; Th. 3.7]). Then our assertion follows from $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regularity and (2.6) (ii).

(iii) To prove the regularity, let $x\notin F$, where F is $\mathscr{T}_i^*$-closed.

At first suppose $\{x\}$ is $\mathscr{T}_i$-closed. Since $x\notin F=\mathscr{T}_i^*\text{-Cl}(F)$, there exists a $\mathscr{T}_i$-g-closed set A such that $F\subset A\subset\mathscr{T}_i\text{-Cl}(A)\subset\{x\}^C$. Then it follows from $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regularity that there exist a $\mathscr{T}_k$-open set U and a $\mathscr{T}_j$-open set V such that $U\cap V=\phi$, $x\in U$ and $\mathscr{T}_i\text{-Cl}(A)\subset V$. By (2.6)(ii), we have $U\in\mathscr{T}_k^*$ and $V\in\mathscr{T}_j^*$.

Suppose $\{x\}$ is not $\mathscr{T}_i$-closed. Then $\{x\}$ is $\mathscr{T}_i^*$-open. Let y be any point of F.

Case 1. Suppose $\{y\}$ is $\mathscr{T}_i$-closed. It follows from $(\mathscr{T}_i;\ \mathscr{T}_j,\ \mathscr{T}_k)$-regularity and (2.6) (ii) that there exist a $\mathscr{T}_k^*$-open set U and a $\mathscr{T}_j^*$-open set V such that $U\cap V=\phi$, $y\in V$ and $x\in U$. Then we have $y\in(\mathscr{T}_j^*\text{-Cl}(x))^C$.

Case 2. Suppose $\{y\}$ is not $\mathscr{T}_i$-closed. Then we have $\{y\}\in\mathscr{T}_i^*$, and hence $\{y\}$ is $\mathscr{T}_j^*$-open by (4.13). Since $x\neq y$, y belongs to $(\mathscr{T}_j^*\text{-Cl}(x))^C$.

By cases 1 and 2 we obtain $F\subset(\mathscr{T}_j^*\text{-Cl}(x))^C$. On the other hand $\{x\}$ is $\mathscr{T}_k^*$-open, by using (4.13). Therefore $(X;\ \mathscr{T}_1^*,\ \mathscr{T}_2^*)$ is $(\mathscr{T}_i^*;\ \mathscr{T}_j^*,\ \mathscr{T}_k^*)$-regular.

REMARK 4.14. When $\mathscr{T}_k=\mathscr{T}_j$ $(j\neq i)$ in Th. 4.12(iii), the condition (4.13) is followed from $T_{1/2}$-condition of $(X;\ \mathscr{T}_i)$.

By setting $\mathscr{T}_1=\mathscr{T}_2$ in Th. 4.12(iii), we have the following

COROLLARY 4.15. (W. Dunham [3; Th. 4.11]). *If $(X;\ \mathscr{T})$ is regular, then $(X;\ \mathscr{T}^*)$ is regular.*

REFERENCES

[1] W. Dunham, *Weakly Hausdorff spaces*, Kyungpook Math. J., 15(1975) 41—50.

[2] W. Dunham, *$T_{2/1}$-spaces*, Kyungpook Math. J., 17 (1977) 161—169.

[3] W. Dunham, *A new closure operator for non-T_1 topologies*, Kyungpook Math. J., 22(1982) 55—60.

[4] J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. (3), 13 (1963) 71—89.

[5] N. Levine, *Generalized closed sets in topology*, Rendiconti del Circ Math Palermo, 19(1970) 89—96.

[6] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., 15(1965) 961—970.

[7] T. Noiri, *On α-continuous functions*, Časopis propěstováni matematiky, roč., 109 (1984) 118—126.

[8] I.L. Reilly, *On Bitopological separation properties*, Nanta Math., 2(1972) 14—25.

[9] I.L. Reilly and M.K. Vamanamurthy, *On α-sets in topological spaces*, Tamkang J. Math. (to appear)

[10] J.D. Weston, *On the comparison of topologies*, J. London Math. Soc. 32(1957), 342—354.

Department of Mathematics,
Fukuoka University of Education,
729 Akama, Munakata, Fukuoka,
811-41, Japan.