Kyungpook Math. J.
Volume 27, Number 2
December, 1987

SEPARATED LOCALES II

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In [4], we have established an equivalence between the category of spatial separated locales and the category of sober T_1-spaces and hence the concept of separated locales is very appropriate for a localic form of T_1-separation axiom.

For frames A, B, the hom-set hom(A, B) of frame homomorphisms on A to B has a natural order.

For an unordered frame X, one has a discrete ordered set hom(X, A) for any frame A. We observe that a frame (=locale) A is separated iff hom$(A, 2)$ has the discrete order. Thus we introduce a concept of $\boldsymbol{A}$-separated frames, namely those frames X whose hom(X, A) have a discrete order fo all $A \in \boldsymbol{A}$ and dually a concept of $\boldsymbol{A}$-discrete frames, where $\boldsymbol{A}$ is class of frames.

It is shown that for a class $\boldsymbol{A}$ of frames, the class $S(\boldsymbol{A})$ of $\boldsymbol{A}$-separated frames is closed under the formation of epi-sinks in **Frm** and the class $D(\boldsymbol{A})$ of $\boldsymbol{A}$-discrete frames is closed under the formation of mono-sources in **Frm**. Furthermore, (S, D) is a Galois connection.

It is also shown that for any class $\boldsymbol{B}$ of frames with $\boldsymbol{A} \subseteq \boldsymbol{B} \subseteq M(\boldsymbol{A})$, $S(\boldsymbol{A}) = S(\boldsymbol{B})$, where $M(\boldsymbol{A})$ is the class of frames which are domains of mono-sources with codomains in $\boldsymbol{A}$.

Using these results, we show that a frame A is separated iff for any spatial frame X, hom(A, X) has a discrete order. Finally we give some interesting examples $S(\boldsymbol{A})$, $D(\boldsymbol{A})$ for various class $\boldsymbol{A}$.

For the terminology, we refer to [6].

1. Discrete order on hom-sets

For frames A, B, its hom-set hom(A, B) has the natural order, namely for $f, g \in$ hom(A, B), $f \leq g$ iff for all $x \in A$, $f(x) \leq g(x)$. It is known [6] that hom(A, B) has directed joins.

In the following, the order on hom(A, B) means the above natural order.

Using the order on hom-sets, we form two classes from a class of frames

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This research is supported by the NSERC, under grant A4809* and a grant from the Korea Science and Engineering Foundation.**

and then investigate their properties.

DEFINITION 1.1. Let $\mathbf{A}$ be a class of frames.

1) A frame X is said to be *$\mathbf{A}$-separated* if for any $A\in\mathbf{A}$, hom (X, A) is a discrete ordered set.

2) A frame X is said to be *$\mathbf{A}$-discrete* if for any $A\in\mathbf{A}$, hom(A, X) is a discrete ordered set.

REMARK 1) The concept of $\mathbf{A}$-separated frames is dual to that of $\mathbf{A}$-discrete frames.

2) A frame X is $\mathbf{A}$-separated iff whenever $u\leq v$ in hom(X, A) $(A\in\mathbf{A})$, one has $u=v$.

In the following, for a class $\mathbf{A}$ of frames, $S(\mathbf{A})$ denotes the class of all $\mathbf{A}$-separated frames and $D(\mathbf{A})$ the class of all $\mathbf{A}$-discrete frames.

PROPOSITION 1.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be classes of frames, then one has,*

1) *If $\mathbf{A}\subseteq\mathbf{B}$, then $S(\mathbf{B})\subseteq S(\mathbf{A})$ and $D(\mathbf{B})\subseteq D(\mathbf{A})$.*

(2) *$\mathbf{B}\subseteq S(\mathbf{A})$ iff $\mathbf{A}\subseteq D(\mathbf{B})$. In other words, the pair(S, D) is a Galois connection.*

PROOF. 1) It is immediate from the definition.

2) Assume $\mathbf{B}\subseteq S(\mathbf{A})$. Take any $A\in\mathbf{A}$ and $B\in\mathbf{B}$. Since $B\in\mathbf{B}\subseteq S(\mathbf{A})$, hom$(B, A)$ has the discrete order and hence $A\in D(\mathbf{B})$. By the exactly same argument, one has the converse.

By the Galois connection (S, D), one has the following. (See [3] for basic properties of Galois connections.)

COROLLARY 1.3. *Let $\mathbf{A}$ be any class of frames, then*

1) $\mathbf{A}\subseteq D(S(\mathbf{A}))$; $\mathbf{A}\subseteq S(D(\mathbf{A}))$.

2) $S(D(S(\mathbf{A})))=S(\mathbf{A})$; $D(S(D(\mathbf{A}))=D(\mathbf{A})$.

Furthermore, for any family $(\mathbf{A}_i)_{i\in I}$ of classes of frames,

3) $S(\bigcup_{i\in I}\mathbf{A}_i)=\bigcap_{i\in I}S(\mathbf{A}_i)$; $D(\bigcup_{i\in I}\mathbf{A}_i)=\bigcap_{i\in I}D(\mathbf{A}_i)$.

THEOREM 1.4. *For any class $\mathbf{A}$ of frames, $S(\mathbf{A})$ is closed under the formation of epi-sinks in* **Frm** *and $D(\mathbf{A})$ is closed under the formation of mono-sources in* **Frm**.

PROOF. Suppose $(f_i : X_i\to X)_{i\in I}$ is an epi-sink in **Frm** such that for all $i\in I$, X_i belongs to $S(\mathbf{A})$. Take any $A\in\mathbf{A}$ and $u, v\in$hom(X_i, A) with $u\leq v$, then for all $i\in I$, $u\circ f_i\leq v\circ f_i$ in $\hom(X_i, A)$. Smce $\hom(X_i, A)$ is a discrete ordered set, $u\circ f_i=v\circ f_i$ for all $i\in I$. Since $(f_i)_{i\in I}$ is an epi-sink, we have $u=v$. Thus X is again a member of $S(A)$. Since D is dual to S, the second half is now immediate from the first half.

COROLLARY 1.5. *For a class* A *of frames,* $S(A)$ *is coproductive and cohereditary in* **Frm**; $D(A)$ *is productive and hereditary in* **Frm**.

REMARK 1.6. Since Loc$=\mathbf{Frm}^{op}$, products of A-separated locals are again A-sublocales and sublocales of an A-separated locales are again A-separated.

NOTATION. For a class A of frames, $M(A)$ denotes the class $\{X\in\mathbf{Frm}\mid$ there is a mono-source $(f_i : X\to A_i)_{i\in I}$ with codomains A_i in A for all $i\in I\}$ and $E(A)$ the class $\{X\in\mathbf{Frm}\mid$ there is an epi-sink$(f_i : A_i\to X)_{i\in I}$ with domains A_i in A for all $i\in I\}$.

Using the above notation, we have the following:

THEOREM 1.7. *Let* A *be a class of frames.*

1) *For any class* A *of frames with* $A\subseteq B\subseteq M(A)$, $S(A)=S(B)$.
2) *For any class* B *of frames with* $A\subseteq B\subseteq E(A)$, $D(A)=D(B)$.

PROOF. 1) Since $A\subseteq B$, we have $S(B)\subseteq S(A)$. Take any $X\in S(A)$ and any $B\in B$, then there is a mono-source $(f_i : B\to A_i)_{i\in I}$ such that for all $i\in I$, $A_i\in A$. For any $u, v\in\hom(X, B)$ with $u\leq v$, we have $f_i\circ u\leq f_i\circ v$ in $\hom(X, A_i)$ ($i\in I$). Since $X\in S(A)$ and $A_i\in A$, $f_i\circ u=f_i\circ v$. Since $(f_i)_{i\in I}$ is a mono-source, we have $u=v$. Thus X belongs to $S(B)$, so that $S(B)=S(A)$.

2) Dual of 1).

For a class A of frames, let $R(A)$ ($Q(A)$) denote the class of all subframes (quotient frames, resp.) of members of A and let $P(A)$ ($C(A)$) denote the class of all product frames (coproduct frames, resp.) of members of A. Then it is immediate that $A\subseteq R(A)$, $P(A)\subseteq M(A)$ and $A\subseteq Q(A)$, $C(A)\subseteq E(A)$ and hence we have the following by the above theorem.

COROLLARY 1.8. *For any class* A *of frames, one has,*

1) $S(A)=S(R(A))=S(P(A))$.
2) $D(A)=D(Q(A))=D(C(A))$.

2. Separated locales

Throughout this section, 2 and 3 will denote the two element chain $\{0, 1\}$

and the three element chain $\{0, 1/2, 1\}$, respectively.

We recall [4] that a frame A is said to be separated if there is no onto frame homomorphism $A \to 3$.

LEMMA 2.1. *A frame A is separated iff A is 2-separated.*

PROOF. Suppose A is separated and there are $u, v \in \hom(A, 2)$ such that $u \leq v$ and $u \neq v$. Let $p = \bigvee u^{-1}(0)$ and $q = \bigvee v^{-1}(0)$. Since $u \leq v$ and $u \neq v$, p and q are distinct prime elements of A with $q \leq p$. We define $h: A \to 3$ as follows: $h(x) = 1$ if $x \nleq p$, $h(x) = 1/2$ if $x \nleq q$ and $x \leq p$, and $h(x) = 0$ if $x \leq q$. Using the fact that p and q are prime elements, one can easily show that h is a frame homomorphism. Since $h(p) = 1/2$, h is onto, so that we have a contradiction.

Conversely, suppose A is not separated but 2-separated. Then there is an onto frame homomorphism $h: A \to 3$. Let x_0 be an element of A with $h(x_0) = 1/2$. Let $u, v: 3 \to 2$ be the characteristic functions of $\{1, 1/2\}$ and $\{1\}$, respectively. It is clear that u and v are frame homomorphisms and that $v \circ h \leq u \circ h$ in $\hom(A, 2)$. Since $u \circ h(x_0) \neq v \circ h(x_0)$, one has also a contradiction. This completes the proof.

REMARK 2.2. For any frame A, there is an isomorphism between $\hom(A, 2)$ and $\mathrm{pt}(A)^{\mathrm{op}}$, where $\mathrm{pt}(A)$ is the set of all prime elements of A. Hence a frame A is separated iff $\mathrm{pt}(A)$ is a discrete orderet set, i.e., for any prime elements p, q of A with $p \leq q$, one has $p = q$.

Since prime elements in a Boolean algebra are precisely coatoms, every complete Boolean algebra is separated.

We recall [6] that a frame A is spatial iff $\hom(A, 2)$ is a mono-source. Let **SFrm** denote the class of all spatial frames.

Using this, one has the following interesting characterizations of separated frames.

THEOREM 2.3. *For a frame A, the following are equivalent:*

1) *A is separated.*
2) *A is 2-separated.*
3) *A is* **SFrm**-*separated.*
4) *A is 3-separated.*

PROOF. By the above lemma, 1) and 2) are equivalent. As mentioned above, $M(2) = \mathbf{SFrm}$. Since 2 is a subframe of 3, $M(2) \subseteq M(3)$. Furthermore, $M(3) \subseteq M(2)$ because $3 \in \mathbf{SFrm} = M(2)$. Thus one has $M(2) = \mathbf{SFrm} = M(3)$; hence by Theorem 1.7, $S(2) = S(M(2)) = S(\mathbf{SFrm}) = S(M(3)) = S(3)$. This completes the proof.

REMARK 2.4. A frame A is unordered iff A is **Frm**-separated. Since $S(\mathbf{Frm}) \subseteq S(\mathbf{SFrm})$, every separated frame is unordered (See Proposition 2.8 in [4]). Let **Sep**(**UO**) denote the class of separated (unordered, resp.) frames. The following is immediate from Theorem 1.4, 2.3 and the above remark (See also Theorem 2.10 in [4]).

COROLLARY 2.5. **Sep** *and* **UO** *are closed under the formation of epi-sinks in* **Frm**.

For the operators S and D, one has the following:

PROPOSITION 2.6. 1) $SD(\mathbf{Sep}) = \mathbf{Sep}$.

2) $SD(\mathbf{UO}) = \mathbf{UO}$.

3) $D(2) = \mathbf{Frm}$.

4) $D(3) = \{1\}$ *and hence* $D(2) \neq D(3)$, *where* 1 *denotes the singleton frame.*

5) $SD(3) = \mathbf{Frm}$.

6) $D(\mathbf{Frm}) = \{1\}$.

7) $D(\mathbf{UO}) = \mathbf{Frm}$.

PROOF. 1) and 2) follows from Corollary 1.3 and the fact that $\mathbf{Sep} = S(\mathbf{SFrm})$ and $\mathbf{UO} = S(\mathbf{Frm})$. Noting that 2 is an initial object of **Frm**, for any $A \in \mathbf{Frm}$, $\hom(2, A)$ is a singleton set; hence $D(2) = \mathbf{Frm}$. Since 1 is a terminal object of **Frm**, $1 \in D(3)$. Suppose a frame A has more than two elements. Define $u, v : 3 \to A$ by $u(1) = u(1/2) = v(1) = 1$ and $u(0) = v(1/2) = v(0) = 0$. Then $u, v \in \hom(3, A)$ with $v \leq u$ but $v \neq u$. Hence $A \notin D(3)$. Thus we have 4). Since 1 is a terminal object of **Frm**, $S(1) = \mathbf{Frm}$, so that $SD(3) = S(1) = \mathbf{Frm}$. 6) follows from the fact that $D(\mathbf{Frm}) = DSD(3) = D(3) = \{1\}$. Finally, $D(\mathbf{UO}) = DS(\mathbf{Frm}) = DSD(2) = D(2) = \mathbf{Frm}$.

REMARK. Since $3 \in \mathbf{SFrm}$, $D(\mathbf{SFrm})$ is contained in $D(3)$; hence $D(\mathbf{SFrm}) = \{1\}$.

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