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DISTRIBUTIVE, STANDARD AND NEUTRAL ELEMENTS IN THE JOINSEMILATTICE OF CONVEX SUBLATTICES OF A LATTICE

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1. Introduction and basic concepts

The purpose of this paper is to generalize the properties of distributive, standard and neutral ideals of a lattice. A generalization of standard ideals is given by Fried and Schmidt in [1], where the concept of a standard convex sublattice is defined and related properties are described. The main difficulty in the generalization work is the lack of a suitable algebra for describing the convex sublattices of a lattice. We shall first introduce an algebra for convex sublattices of a lattice and thereafter consider distributive, standard and neutral convex sublattices by means of the properties of the algebra.

A χ_{lub}-lattice $H=(H, \vee, \wedge)$ is a joinsemilattice, where $a\vee b=\mathrm{lub}\{a, b\}$, the least upper bound of a and b, for every two elements $a, b\in H$, and $a\wedge b=\mathrm{glb}\{a, b\}$, the greatest lower bound of a and b, when the set $\mathrm{lb}\{a, b\}$ of lower bounds of a and b is nonempty. If $\mathrm{lb}\{a, b\}=\phi$, we put $a\wedge b=a\vee b$. Thus the operation $\vee$ behaves like the corresponding operation in a joinsemilattice, i.e. it is associative and $c\leqslant a$ and $d\leqslant b$ imply $c\vee d\leqslant a\vee b$. Unfortunately, $c\leqslant a$ and $d\leqslant b$ need not imply $c\wedge d\leqslant a\wedge b$, and $\wedge$ need not be associative. On the other hand, $a\wedge a=a$ and $a\wedge b=b\wedge a$ for all $a, b\in H$. As easily seen, every lattice is also a χ_{lub}-lattice, but every joinsemilattice S need not be a χ_{lub}-lattice, because the property $\mathrm{lb}\{a, b\}\neq\phi$ in S need not imply the existence of an element $\mathrm{glb}\{a, b\}$ in S. A χ_{lub}-lattice H is called distributive (modular) if the conditions D_1 and D_2 (M_1 and M_2) below hold:

D_1 $a\wedge(b\vee c)\leqslant(a\wedge b)\vee(a\wedge c)$ for all $a, b, c\in H$;

D_2 $a\vee(b\wedge c)\geqslant(a\vee b)\wedge(a\vee c)$ for all $a, b, c\in H$;

M_1 $a\wedge(b\vee(c\wedge a))\leqslant(a\wedge b)\vee(a\wedge c)$ for all $a, b, c\in H$;

M_2 $a\vee(b\wedge(c\vee a))\geqslant(a\vee b)\wedge(a\vee c)$ for all $a, b, c\in H$.

Clearly every distributive χ_{lub}-lattice is also modular. Note that the equality sign need not always hold in D_1, D_2, M_1 and M_2. Appropriate examples one can find by considering e.g. finite trees.

Let $Csub\ (L)$ be the set of nonempty convex sublattices of a lattice L and $A, B \in Csub(L)$. As well known, the least convex sublattice of L containing A and B is $A \vee B = \{x \mid x \in L,\ a_1 \wedge b_1 \leq x \leq a_2 \vee b_2$ for some $a_1,\ a_2 \in A$ and $b_1,\ b_2 \in B\}$. Moreover, if $A \cap B \neq \phi$, then there is a greatest convex sublattice of L contained in A and B, namely $A \wedge B = A \cap B$. Now, by putting $A \wedge B = A \vee B$ for A and B with $A \cap B = \phi$, we see that the convex sublattices of a lattice constitute a χ_{lub} -lattice $Csub(L)$, where $A \vee B$ is given above, $A \wedge B = A \cap B$ when $A \cap B \neq \phi$ and otherwise $A \wedge B = A \vee B$. Note that when $A \cap B \neq \phi$ and $w \in A \cap B$, then $a \wedge b \in A \wedge B$ for $a,\ b \geq w$ and $a \vee b \in A \wedge B$ for $a,\ b \leq w$, where a is from A and b from B. As well known, every ideal of L is also a convex sublattice of L. If I and J are two ideals of L, then $I \wedge J = I \cap J$ in the lattice $I(L)$ of all ideals of L and $I \vee J = \{x \mid x \in L,\ x \leq i \vee j$ for some $i \in I$ and $j \in J\}$ in $I(L)$. Thus the meet and join of two ideals in $I(L)$ and $Csub(L)$ coincide and we shall use a single sign $\vee$ ($\wedge$) for the join (the meet) in $I(L)$ as well as in $Csub(L)$.

An element $A \in Csub(L)$ is

(1) ***distributive*** if and only if $A \vee (X \wedge Y) \geq (A \vee X) \wedge (A \vee Y)$ for all $X,\ Y \in Csub(L)$;

(2) ***standard*** if and only if it is distributive and (3) and (4) hold, where

(3) when $A \cap X \neq \phi$, then $A \wedge X = A \wedge Y$ and $A \vee X = A \vee Y$ imply $X = Y$;

(4) when $A \cap X = \phi$, then $(A] \wedge (X] = (A] \wedge (Y]$ and $(A] \vee (X] = (A] \vee (Y]$ imply $(X] = (Y]$, $X,\ Y \in Csub(L)$ and $(X] = \{z \mid z \in L$ and $z \leq x$ for some $x \in X\}$;

(5) ***neutral*** if and only if it is standard and dually distributive.

2. Distributive convex sublattices

In this section we shall describe distributive convex sublattices of a lattice L. At first we write a lemma, the proof of which is obvious and hence omitted.

LEMMA 1. *Let* $A, X, Y \in Csub(L)$. *If* $X \cap Y = \phi$, *then* $A \vee (X \wedge Y) \geq (A \vee X) \wedge (A \vee Y)$.

The following theorem shows a connection between convex sublattices, ideals and dual ideals of L.

THEOREM 1. *Let* $A \in Csub(L)$. *Then* A *is distributive in* $Csub(L)$ *if and only if* $(A]$ *is distributive in* $I(L)$ *and* $[A)$ *is distributive in the lattice* $D(L)$ *of dual ideals of* L.

PROOF. Assume first that A is distributive in $Csub(L)$. We shall show the distributivity of $(A]$ in $I(L)$ only; the proof for $[A)$ is analogous and hence omitted.

Let $(A]=\{x \mid x\leqslant a,\ a\in A\}$. Because A is a sublattice of L, $x_1\vee x_2\leqslant a_1\vee a_2\in A$ for any two elements $x_1,\ x_2\in(A]$, and thus $(A]$ is an ideal of L. Let $X,Y\in I(L)\subset Csub(L)$. Because A is distributive in $Csub(L)$, $A\vee(X\wedge Y)\geqslant(A\vee X)\wedge(A\vee Y)$ in $Csub(L)$. If $z\in((A]\vee X)\wedge((A]\vee Y)$, then $z\in(A]\vee X,\ (A]\vee Y$, and so $z\leqslant a_1\vee x_1,\ a_2\vee y_2$, where $a_1,\ a_2\in A,\ x_1\in X,\ y_2\in Y$ and $(a_1\vee x_1)\wedge(a_2\vee y_2)\in(A\vee X)\wedge(A\vee Y)$. Because $A\vee(X\wedge Y)\geqslant(A\vee X)\wedge(A\vee Y)$, $(a_1\vee x_1)\wedge(a_2\vee y_2)\in A\vee(X\wedge Y)$. But then $a_3\vee(x_3\wedge y_3)\geqslant(a_1\vee x_1)\wedge(a_2\vee y_2)\geqslant z$, and thus $z\in(A]\vee(X\wedge Y)$. This shows that $(A]\vee(X\wedge Y)\geqslant((A]\vee X)\wedge((A]\vee Y)$, and the distributivity of $(A]$ in $I(L)$ follows.

Conversely, let $A\in Csub(L)$, $(A]$ be distributive in $I(L)$ and $[A)$ distributive in $D(L)$. We shall show the distributivity of A in $Csub(L)$.

Let $z\in(A\vee X)\wedge(A\vee Y)$, where X and Y are two arbitrary elements of $Csub(L)$. Because $\phi\neq A\subset(A\vee X)\wedge(A\vee Y)$, then $z\in A\vee X,\ A\vee Y$, whence $a_1\wedge x_1,\ a_2\wedge y_2\leqslant z\leqslant a_3\vee x_3,\ a_4\vee y_4$ with $a_1,\ a_2,\ a_3,\ a_4\in A,\ x_1,\ x_3\in X$ and $y_2,\ y_4\in Y$. According to Lemma 1 we may assume that $X\cap Y\neq\phi$. When $w\in X\wedge Y$, we may clearly choose the elements x_i and y_j above so that $x_1,\ y_2\leqslant w\leqslant x_3,\ y_4$. Obviously $(a_3\vee x_3)\wedge(a_4\vee y_4)$ belongs to the ideal $((A]\vee(X])\wedge((A]\vee(Y])$, to which thus z also belongs. According to the distributivity of $(A]$ in $I(L)$, $((A]\vee(X])\wedge((A]\vee(Y])=(A]\vee((X]\wedge(Y])$, whence $(a_3\vee x_3)\wedge(a_4\vee y_4)\leqslant a'\vee(x'\wedge y')$ with $a'\in(A]$, $x'\in(X]$ and $y'\in(Y]$. We can clearly choose new elements a, x and y from A, X and Y, respectively, such that $a'\vee(x'\wedge y')\leqslant a\vee(x\wedge y)$, $a'\leqslant a$, $x'\leqslant x,\ w$ and $y'\leqslant w, y$. Consequently, $z\leqslant a\vee(x\wedge y)\in A\vee(X\wedge Y)$. Similarly, $z,\ (a_1\wedge x_1)\vee(a_2\wedge y_2)\in([A)\vee[X))\wedge([A)\wedge[Y))$, and by using the distributivity of $[A)$ in $D(L)$ and the dual argumentation, we obtain that $a''\wedge(x''\vee y'')\in A\vee(X\wedge Y)$ with $a''\wedge(x''\vee y'')\leqslant z$. Accordingly, $a''\wedge(x''\vee y'')\leqslant z\leqslant a\vee(x\wedge y)$, where both limits are from $A\vee(X\wedge Y)$, whence $z\in A\vee(X\wedge Y)$. The results above and Lemma 1 imply now that $A\vee(X\vee Y)\geqslant(A\vee X)\wedge(A\vee Y)$ for all $X,Y\in Csub(L)$, which proves the distributivity of A in $Csub(L)$.

As a corollary we can write

COROLLARY 1. *Let $a\leqslant b$ in L. The interval $[a,\ b]\in Csub(L)$ is distributive in $Csub(L)$ if and only if b is distributive and a dually distributive in L. Moreover, $\{b\}$ is distributive in $Csub(L)$ if and only if b is distributive as well* *as dually distributive in L.*

3. Standard elements of Csub(L).

The standardness of an element in a lattice has many equivalent definitions. Although one can modify these definitions for $Csub(L)$, the equivalence need not hold any more. As an example we consider the condition

(6) $X\wedge(A\vee Y)\leqslant(X\wedge A)\vee(X\wedge Y)$ for all $X, Y\in Csub(L)$.

At first we prove

LEMMA 2. *Let* $A, X, Y\in Csub(L)$. *The inequality* $X\wedge(A\vee Y)\leqslant(X\wedge A)\vee(X\wedge Y)$ *holds if* $X=L$ *or* $A\leqslant Y$ *or* $X\cap A=\phi$ *or* $X\cap Y=\phi$ *or* $X\cap(A\vee Y)=\phi$.

PROOF. If $X=L$, then $X\wedge(A\vee Y)=A\vee Y=(X\wedge A)\vee(X\wedge Y)$. If $A\leqslant Y$, then $X\wedge(A\vee Y)=X\wedge Y\leqslant(X\wedge A)\vee(X\wedge Y)$. If $X\cap A=X\cap Y=\phi$, then $(X\wedge A)\vee(X\wedge Y)=(X\vee A)\vee(X\vee Y)=X\vee A\vee Y\geqslant X\wedge(A\vee Y)$. If $X\cap A=\phi$ and $X\cap Y\neq\phi$, then $(X\wedge A)\vee(X\wedge Y)=X\vee A\geqslant X\geqslant X\wedge(A\vee Y)$. If $X\cap A\neq\phi$ and $X\cap Y=\phi$, then $(X\wedge A)\vee(X\wedge Y)=X\vee Y\geqslant X\geqslant X\wedge(A\vee Y)$. If $X\cap(A\vee Y)=\phi$, then $X\cap A=X\cap Y=\phi$, and this case is proved above.

Now we can prove

THEOREM 2. *Let* $A\in Csub(L)$ *be standard. Then* (6) *holds.*

PROOF. Let $B=X\wedge(A\vee Y)$ and $C=(X\wedge A)\vee(X\wedge Y)$, and according to Lemma 2 we may assume that $X\cap A\neq\phi\neq X\cap Y$, $X\cap(A\vee Y)\neq\phi$, $X\neq L$ and $A\nleqq Y$. We show first that $A\wedge B=A\wedge C$. After showing $A\vee B=A\vee C$ we can conclude by (3) that $B=C$. This and Lemma 2 prove then the validity of (6).

Now $A\wedge X=A\cap X$, $X\cap A\subset C$ and $X\cap A\subset A\cap C=A\wedge C$. Further, $A\wedge B=A\cap B=A\cap(X\cap(A\vee Y))=(A\cap X)\cap(A\vee Y)=A\cap X=A\wedge X$. Moreover, $X\cap A$, $X\cap Y\subset X\cap(A\vee Y)=B$, whence $C\subset B$ and $A\cap C\subset A\cap B$. By combining these results we obtain $A\wedge X\leqslant A\wedge C\leqslant A\wedge B\leqslant A\wedge X$, and thus $A\wedge C=A\wedge B$, where $A\cap B\neq\phi\neq A\cap C$.

Secondly we prove $A\vee B=A\vee C$. Clearly $A\vee(X\cap Y)\subset A\vee X$, $A\vee Y$, whence $A\vee(X\cap Y)\subset(A\vee X)\cap(A\vee Y)$. Thus, under the assumptions made above, the distributivity of A in $Csub(L)$ implies the equality $A\vee(X\wedge Y)=(A\vee X)\wedge(A\vee Y)$. Now $A\vee B=A\vee(X\wedge(A\vee Y))=(A\vee X)\wedge(A\vee Y)=A\vee(X\wedge Y)=A\vee(X\wedge A)\vee(X\wedge Y)=A\vee C$. This completes the proof.

The converse does not hold; this will be shown by an example after the next theorem.

THEOREM 3. *Let $A\in Csub(L)$. If A is standard in $Csub(L)$, then $(A]$ is standard in $I(L)$.*

PROOF. We shall show that $(A]$ is distributive in $I(L)$, and $(A]\wedge X=(A]\wedge Y$ and $(A]\vee X=(A]\vee Y$ imply $X=Y$ for all $X,Y\in I(L)$, from which the standardness of $(A]$ in $I(L)$ follows [2, Theorem Ⅲ.3.5].

Because A is distributive in $Csub(L)$, the distributivity of $(A]$ in $I(L)$ follows from Theorem 1. Let now $(A]\wedge X=(A]\wedge Y$ and $(A]\vee X=(A]\vee Y$ for some $X,Y\in I(L)$. If $A\cap X\neq\phi$, then also $A\cap Y\neq\phi$, because $(A]\wedge X=(A]\wedge Y$ and $(A]\cap X$ contains at least one element from A. By similar argument we see that $A\cap X=A\cap Y$, and thus in this case $A\wedge X=A\wedge Y$ in $Csub(L)$. When $X\in I(L)$, then $A\vee X=(A]\vee X$ in $Csub(L)$, whence the equation $A\vee X=A\vee Y$ follows from $(A]\vee X=(A]\vee Y$. Because A is standard in $Csub(L)$, (3) implies now $X=Y$. When $A\cap X=\phi$, then $X=Y$ follws by (4) from $(A]\wedge X=(A]\wedge Y$ and $(A]\vee X=(A]\vee Y$. This completes the proof.

Now we show that (6) does not imply the standardness of an element $A\in Csub(L)$. Let L be the well known least modular and nondistributive lattice with elements $0<a,b,c<1$. We put $A=\{a\}$ and show that $X\wedge(A\vee Y)\leqslant(X\wedge A)\vee(X\wedge Y)$ for all $X,Y\in Csub(L)$. According to Lemma 2 we may assume that $X\neq L$, $A\nleqq Y$, $X\cap A\neq\phi\neq X\cap Y$ and $X\cap(A\vee Y)\neq\phi$. $X\cap A\neq\phi$ implies that $a\in X$, and $A\nleqq Y$ that $a\notin Y$. Since $X=\{a\}$ contradicts $X\cap Y\neq\phi$, we have $X=(a]$ or $[a)$. If $X=(a]$, then $X\cap Y=\{0\}$, whence $(X\wedge A)\vee(X\wedge Y)=\{a\}\vee\{0\}=X\geqslant X\wedge(A\vee Y)$. If $X=[a)$, then $X\wedge Y=\{1\}$, whence $(X\wedge A)\vee(X\wedge Y)=\{a\}\vee\{1\}=X\geqslant X\wedge(A\vee Y)$. Hence A satisfies (6). But $(a]$ is not standard in $I(L)$, and thus by Theorem 3 A is certainly not standard in $Csub(L)$.

We call an element $A\in Csub(L)$ *double standard* if A is distributive in $Csub(L)$ and satisfies (3), (4) and (7), where

(7) when $A\cap X=\phi$, then $[A)\wedge[X)=[A)\wedge[Y)$ and $[A)\vee[X)=[A)\vee[Y)$ imply $[X)=[Y)$ for all $X,Y\in Csub(L)$.

Now we can prove

THEOREM 4. *Let $A\in Csub(L)$. A is double standard in $Csub(L)$ if and only if $(A]$ is standard in $I(L)$ and $[A)$ standard in $D(L)$.*

PROOF. Let A be double standard. The standardness of $(A]$ in $I(L)$ is already proved in Theorem 3. The standardness of $[A)$ in $D(L)$ can be proved dually, and in the dual proof (4) is substituted by (7), as easily seen. Thus we concentrate on the converse proof, only.

Let $(A]$ be standard in $I(L)$, $[A)$ standard in $D(L)$ and $X, Y \in Csub(L)$. Because $(A]$ is then distributive in $I(L)$ and $[A)$ distributive in $D(L)$, A is distributive in $Csub(L)$ by Theorem 1. Thus it remains to show the validity of (4), (7) and (3). If $A \cap X = \phi$, then $(A] \wedge (X] = (A] \wedge (Y]$ and $(A] \vee (X] = (A] \vee (Y]$ imply $(X] = (Y]$ because $(A]$ is standard in $I(L)$ [2, Thorem Ⅲ.3.5]. The validity of (7) is proved dually. Assume now that $A \cap X \neq \phi$, $A \wedge X = A \wedge Y$ and $A \vee X = A \vee Y$. As one can easily see, these equations imply $(A] \wedge (X] = (A] \wedge (Y]$, $(A] \vee (X] = (A] \vee (Y]$, $[A) \wedge [X) = [A) \wedge [Y)$ and $[A) \vee [X) = [A) \vee [Y)$. Because $(A]$ is standard in $I(L)$, the first two equations imply $(X] = (Y]$, and because $[A)$ is standard in $D(L)$, the remaining two equations imply $[X) = [Y)$. But then $X = (X] \cap [X) = (Y] \cap [Y) = Y$, which proves (3). Accordingly, A is double standard in $Csub(L)$, and the theorem follows.

Because L is standard in $I(L)$ as well as in $D(L)$, $[I) = L$ is standard in $D(L)$ for every $I \in I(L)$ and $(D] = L$ is standard in $I(L)$ for every $D \in D(L)$. Thus by Theorem 4 every standard ideal I (standard dual ideal D) of L is double standard in $Csub(L)$. The convex sublattice $\{b\}$ is standard in $Csub(L)$ if b is standard and dually distributive in L. Indeed, the standardness and the dual distributivity of b in L imply the distributivity of $\{b\}$ in $Csub(L)$, and (4) holds by the standardness of b in L. If $\{b\} \cap X \neq \phi$, then $b \in X, Y$, and thus $X = \{b\} \vee X = \{b\} \vee Y = Y$, which proves (3). Now we can write a corollary

COROLLARY 2. *Every standard ideal (dual ideal) of L is double standard in $Csub(L)$. An interval $[a, b]$ is double standard in $Csub(L)$ if and only if b is standard and a dually standard in L. $\{b\}$ is double standard in $Csub(L)$ if and only if b is neutral (i.e. standard and dually standard) in L, and $\{b\}$ is standard in $Csub(L)$, if b is standard and dually distributive in L.*

As well known, in a modular lattice L an ideal (a dual ideal) is distributive if and only if it is standard [2, Theorem Ⅲ.2.6]. This and Theorems 1 and 4 imply

THEOREM 5. *Let L be a modular lattice and $A \in Csub(L)$. A is distributive in $Csub(L)$ if and only if A is standard in $Csub(L)$. Moreover, A is standard in $Csub(L)$ if and only if A is double standard in $Csub(L)$.*

4. Neutral convex sublattices

At first we like to show a connection between the neutrality of A in $Csub(L)$ and the neutrality of $(A]$ in $I(L)$.

THEOREM 6. *Let $A \in Csub(L)$. If A is neutral in $Csub(L)$, then $(A]$ is neutral in $I(L)$.*

PROOF. When A is neutral, it is also standard, and thus by Theorem 3 we know that $(A]$ is standard in $I(L)$. The neutrality of $(A]$ in $I(L)$ is proved by showing the dual distributivity of $(A]$ in $I(L)$ [2, Theorem Ⅲ.3.6]. The dual proof of Theorem 1 shows that the dual distributivity of A in $Csub(L)$ implies the dual distributivity of $(A]$ in $Csub(L)$, and thus the neutrality of A in $Csub(L)$ implies the neutrality of $(A]$ in $I(L)$.

By modifying the definition of double standardness, the concept of double neutrality in $Csub(L)$ can be defined and an analogy of Theorem 4 proved. This generalizaion is obvious, and hence we omit it. Also an analogy of Corollary 2 as well as that of Theorem 5 can be easily presented.

As a last observation of this paper we like to give another immediate generalization. When L is a distributive lattice then $(A]$ is distributive in $I(L)$ as well as $[A)$ in $D(L)$ for every $A \in Csub(L)$. According to Thorem 1 and its dual we see that then $Csub(L)$ is a distributive χ_{lub}-lattice. Conversely, when $Csub(L)$ is a distributive χ_{lub}-lattice, then $I \vee (J \wedge K) \geqslant (I \vee J) \wedge (I \vee K)$ for all three ideals I, J, K of L in $Csub(L)$. Hence $I(L)$ is a distributive lattice and thus L, too. Accordingly we can write

THEOREM 7. *A lattice L is distributive if and only if $Csub(L)$ is a distributive χ_{lub}-lattice.*

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