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ON GENERALIZED FLOQUET SYSTEMS II

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Abstract: Consider the system (i) $x'=Ax$. Let Φ be its fundamental matrix solution. If there is $w>0$ such that $A(t+w)-A(t)$ commutes with Φ for all t, then we call this system a "generalized" Floquet system or a "G.F. system". We show that $A(t+w)-A(t)=B_1=$constant if and only if $A(t)=C+B_1t/w+Q(t)$, Q is periodic of period $w>0$. For this $A(t)$ we prove that if all eigenvalues of B_1 have negative real parts, then the origin is asumptotically stable. We find a growth condition for a continuous $D(t)$ which guarantees that all solutions of $z'=[A(t)+D(t)]z$ are bounded if all solutions of the G.F. system (i) are bounded. Combining the foregoing results yield a class of perturbed G.F. operators all of whose solutions are bounded.

1. Introduction

Floquet's theorem states that for the linear system

$$x'=A(t)x, \quad -\infty<t<\infty \tag{1}$$

where x is an n-dimensional column vector, $A(t)$ is $n\times n$ matrix whose elements are continuous functions for all t, if there exists $w>0$ such that

$$A(t+w)=A(t) \tag{2}$$

for all t, then there exists a nonsingular matrix C such that for all t, the following equality is valid

$$\Phi(t+w)=\Phi(t) \tag{3}$$

where $\Phi(t)$ is any fundamental matrix of system (1). It follows that there exists a matrix $P(t)$ and a constant nonsingular matrix R such that for all t,

$$\Phi(t)=P(t)e^{Rt}, \quad P(t+w)=P(t)$$

The authors (c.f. [1]) considered the case $A(t+w)\neq A(t)$ and used the notations:

$$B(t, w)=A(t+w)-A(t), \quad [U, V]=UV-VU$$

They gave the following definition:

DEFINITION. The system (1) with $B(t, w)=A(t+w)-A(t)$ is called *a gene-*

ralized Floquet system, or G. F. system if there exists $w>0$, such that

$$[B(t,\ w),\ \Phi(t)]=0,\ t\in(-\infty,\ \infty)$$

They studied the case in which $B(t,\ w)=B_1$, where B_1 is a constant matrix. The general form of $A(t)$ such that $B(t,\ w)=B_1$ is

$$A(t)=C+(B_1/w)t+Q(t) \tag{4}$$

where C and B_1 are $n\times n$ constant nonsingular matrices and $Q(t)$ is a periodic matrix of period w.

The fundamental matrix $\Psi(t,\ w)$ of the system

$$y'=B(t,\ w)y$$

takes the form $\Psi(t,\ w)=\exp(B_1t)$.

They proved the following relations:

$$\Phi(t,\ w)=\Phi(t).\exp(B_1t).\Phi(w)$$

$$\Phi(t+nw)=\Phi(t)\left\{\Phi(w)\exp\left[B_1\left[t+\frac{(n-1)}{2}w\right]\right]\right\}^n \tag{5}$$

and

$$P(t+nw)=P(t)\exp\left\{B_1\left[nt+n\frac{(n-1)}{2}w\right]\right\} \tag{6}$$

provided

$$\Phi(t)=P(t)\exp(Rt),\ \ [R,\ B(t,\ w)]=0$$

2. Asymptotic properties

We shall study the properties of the solutions of the system (1) with $A(t)=C+\frac{B_1}{w}t+P(t)$, and the perturbed system

$$x(t)=[A(t)+D(t)]x$$

THEOREM 1. *Suppose that x satisfies*

$$x'=\left(C+\frac{B_1}{w}t+P(t)\right)x, \tag{7}$$

where C and B_1 are constant matrices, P is periodic with period w and all eigenvalues of B_1 have negative real parts. If B_1 and C commute, then $\lim_{t\to\infty}x(t)=0$.

PROOF. Consider the system

$$Z'=\left(C+\frac{B_1}{w}t\right)Z. \tag{8}$$

The fact that B_1 and C commute implies that

$$\frac{d}{dt}e^{\left(Ct+\frac{B_1}{2w}t^2\right)}=\left(C+\frac{B_1}{w}t\right)e^{\left(Ct+\frac{B_1}{2w}t^2\right)}.$$

Hence,

$$Z(t)=e^{Ct+\frac{B_1}{2w}t^2}=e^{Ct}.e^{\frac{B_1}{2w}t^2}, \tag{9}$$

is a fundamental matrix of (8). Then,

$$Z(t)Z^{-1}(s)=e^{\{C(t-s)\}}.e^{\left\{\frac{B_1}{2w}(t^2-s^2)\right\}}.$$

Let $t-s\geq 0$. Then,

$$\|Z(t)Z^{-1}(s)\|\leq\|e^{C(t-s)}\|\ \|e^{\frac{B_1}{2w}(t^2-s^2)}\|$$
$$\leq e^{\|C\|(t-s)}\|e^{\frac{B_1}{2w}(t^2-s^2)}\| \tag{10}$$

The eigenvalues of B_1 all have negative real parts imply that any solution u of $u'=(B_1/2w)u$ satisfies $\|u(\alpha)\|=\|e^{(B_1\alpha/2w)}\|\leq Me^{-k\alpha}$, for some M, $k>0$ and all $\alpha\geq 0$. Put $\alpha=t^2-s^2>0$, we get:

$$\|e^{\frac{B}{2w}(t^2-s^2)}\|\leq M.e^{-k(t^2-s^2)}. \tag{11}$$

Then, (8) takes the form:

$$\|Z(t)Z^{-1}(s)\|\leq e^{\|C\|(t-s)}.M.e^{-k(t^2-s^2)} \tag{12}$$

Let x be a solution of (7) and let z be the solution of (8) such that $z(0)=x(0)$. Then,

$$x(t)=z(t)+\int_0^t z(t)z^{-1}(s)P(s)x(s)ds,$$

$$\|x(t)\|\leq\|z(t)\|+\int_0^t\|z(t)z^{-1}(s)\|\|P(s)\|\ |x(s)|ds. \tag{13}$$

From (9) and (11), it follows:

$$\|z(t)\|\leq M\|z(0)\|\exp\{\|C\|t-kt^2\}. \tag{14}$$

From (12), (13) and (14), we get:

$$\|x(t)\|\leq M\|z(0)\|\exp\{\|C\|t-kt^2\}+$$
$$+\int_0^t M\exp\{\|C\|(t-s)\}.\exp\{-k(t^2-s^2)\}\ \{\|p(s)\|\ \|x(s)\|ds,$$

$$\|x(t)\|\exp\{\|C\|(t)+kt^2\}\leq M\|z(0)\|+\int_0^t M\exp\{\|C\|(-s)+$$
$$+ks^2\}\|p(s)\|\ \|x(s)\|ds.$$

Applying the well known result, Gronwall-Bellman inequality, we have

$$\|x(t)\|\exp\{\|C\|(-t)+kt^2\}\leq M\|z(0)\|\exp\left\{\int_0^t p(s)ds\right\}. \tag{15}$$

Let ρ be the maximum of the periodic function $\|p(s)\|$. Then,

$$\|x(t)\|\leq M\|z(0)\|\exp\{\rho t+|C|t-kt^2\} \tag{16}$$

Thus, $\lim_{t\to\infty}\|x(t)\|=0$. This completes the proof.

REMARK. The only place that the periodicity of P was used was when in going from (15) to (16), we used the fact that $P(t)$ is bounded. Hence, we have the more general result:

THEOREM 2. *The statement of theorem* 1 *is valid if we replace P by any continuous bounded matrix.*

REMARK. In this section we used the statement $B(t,\ w)=A(t+w)-A(t)$ is independent of t implies

$$A(t)=C+\frac{B}{w}t+P(t)$$

where C and B are constant matrices and P is periodic with period w. To prove this statement, let $A(t)$ be any smooth matrix such that for some w, we have

$$B(t,\ w)=A(t+w)-A(t) \tag{*}$$

is independent of t. Then, $dB(t,\ w)/dt=0$. Hence, dA/dt is periodic with period w. Consider the Fourier series of dA/dt. If we integrate this series, we get:

$$A(t)=C+Dt+P(t),$$

where C and D are constants, P is periodic with period w. This and (*) imply $B(t,\ w)=Dw$, and this completes the proof.

Consider the system

$$y'=A(t)y, \tag{17}$$

and the corresponding perturbed system

$$x'=[A(t)+D(t)]x, \tag{18}$$

where A is defined by (4) and D is a $n\times n$ continuous matrix on $0\leq t<\infty$.

Our hypotheses are:

(i) The system (1) is a G.F. system with $B(t,\ w)=B_1$, where B_1 is a constant matrix.

(ii) Each eigenvalue of B_1 has negative real part and consequently

$$\|e^{B_1 t}\|\leq Me^{-kt}$$

for some M, $k>0$ and all $t\geq 0$.

(iii) The fundamental matrix solution $\Phi(t)$ of the system (13) and its inverse are bounded, i.e.,

$$M_1=\max_{0\leq t\leq w}\|\Phi(t)\|,\quad M_2=\max_{0\leq t\leq w}\|\Phi^{-1}(t)\|$$

We will now exploit (5) to make our main applications.

THEOREM 3. *If*

$$y'=A(t)y=\Big(C+\frac{B_1}{w}t+Q(t)\Big)y, \tag{17}$$

is a G.F. system and every solution of it is bounded, then every solution of

$$x'=[A(t)+D(t)]\,x \tag{18}$$

is bounded also provided

$$\sum_{k=1}^{\infty}\|\Phi(w)\|^{-(k-1)w}\|\exp\Big\{-B_1\frac{(k-1)(k-2)}{2}w\Big\}\|\int_{v=0}^{w}\|\exp -B_1(k-1)v\|\ \|D(v+(k-1)w\|\ dv<\infty. \tag{19}$$

Before proving the theorem we need to prove the following claim:

CLAIM. $\int_0^\infty\|\Phi^{-1}(s)\|\ \|D(s)\|ds<\infty.$

PROOF. Given $t>0$, let m be an integer such that $mw\geq t$. Then,

$$\int_0^t\|\Phi^{-1}(s)\|\ \|D(s)\|ds\leq\sum_{k=1}^{m}\int_{(k-1)w}^{kw}\|\Phi^{-1}(s)\|\ \|D(s)\|ds. \tag{20}$$

Define a new variable of integration by $v=s-(k-1)w$, so that (20) takes the form

$$\int_0^t\|\Phi^{-1}(s)\|\ \|D(s)\|ds\leq\sum_{k=1}^{m}\int_{v=0}^{w}\|\Phi^{-1}(v+(k-1)w)\|\times\|D(v+(k-1)w)\|dv. \tag{21}$$

Since equation (17) is G.F. system, hence (5) and $[\Phi,\ B_1]=0$ imply

$$\Phi^{-1}[v+(k-1)w]=\Phi^{-1}(v)\,[\Phi(w)]^{-(k-1)w}\exp\{-B_1[(k-1)v+\frac{(k-1)(k-2)}{2}w]\}. \tag{22}$$

By hypothesis $\|\Phi^{-1}(v)\|$ is bounded. Hence, if we substitute (22) into the right member of (21), and then apply (19) the claim is proved.

PROOF OF THEOREM 3. Now, represent the solution of (18) with the initial condition $x(0)=c$ by the well known relation

$$x(t)=y(t)+\int_0^t \Phi(t)\Phi^{-1}(s)D(s)x(s)ds, \tag{23}$$

where y is the solution of (17) with initial condition $y(0)=C$. We know that y is bounded, say $\|y(t)\|<a$. Thus, after taking the norm of both members of (23) and applying the Gronwall-Bellman lemma, we have:

$$\|x(t)\|\leq a \exp\{\|\Phi(t)\|\int_0^t \|\Phi^{-1}(s)\| \; \|D(s)\|ds\}.$$

To this inequality, we apply the claim together with the hypothesis that $\|\Phi(t)\|$ is bounded. This proves that $x(t)$ is bounded and completes the proof of the theorem.

THEOREM 4. *If all of the eigenvalues of B_1 have negative real parts, Q has a period $w>0$, and* (17) *is a G.F. system, then,* $\lim_{t\to\infty} y(t)=0$ *for each solution $y(t)$ of* (17).

PROOF. Since the system (17) is a G.F. system and hence $[\Phi, B_1]=0$, then equation (5) implies

$$\|\Phi(t+nw)\|\leq\|\Phi(t)\| \{\Phi(w) \; \|\exp B_1 [t+(n-1)w/2]\|\}^n.$$

Using hypothesis (ii) we have:

$$\|\Phi(t+nw)\|\leq\|\Phi(t)\| \{\|\Phi(w)\| M^2 \exp(-k) \; [t+(n-1)w/2]\}^n.$$

Let $\|\Phi(t)\|\leq M_1$ for all $t\in[0, w]$. Then, for all $t\in[0, w]$,

$$\|\Phi(t+nw)\|\leq M_1 \{M_1 M^2 \exp(-k)(n-1)w/2\}^n.$$

Hence, given $\varepsilon>0$, there exists N such that $n>N$ implies $\|\Phi(t+nw)\|<\varepsilon$ for all $t\in[0, w]$. Since $w>0$, the proof is complete.

COROLLARY. *If Q has period $w>0$, $B_1=bU$, $b<0$ and* (19) *holds, then every solution of* (18) *is bounded.*

PROOF. When $B_1=bU$, system (17) is a G.F. system, hence, theorem implies every solution of (17) is bounded. We can then apply theorem 2 and the proof is complete.

Occasionlly, we encounter a system of the form

$$x'=A(t)x+f(t, x), \tag{24}$$

where $f\in C[J\times R^n, R^n]$, $J=[0, \infty)$. We assume that

$$\|f(t, x)\|\leq\alpha(t)\|x\|. \tag{25}$$

where $\alpha(t)$ is a positive function on J, and

$$\int_0^{\infty} \alpha(t)dt < \infty. \tag{26}$$

THEOREM 5. *Assume that the fundamental matrix* Φ *satisfies the hypothesis* (*iii*) *and* $f(t, x)$ *satisfies* (25) and (26).

If y *is a solution of* (1) *with* $y(t_0)=x_0$ *such that*

$$\lim_{t\to\infty} y(t)=0, \text{ then } \lim_{t\to\infty} x(t)=0.$$

PROOF. The proof is similar to that of theorem 3 and so will be omitted.

GENERAL REMARK. Consider the case where $B(t, w)$ is of the form:

$$B(t, w)=\sum_{i=0}^{N} B_i(t)t^i \tag{27}$$

where B_i, $i=0, 1, 2, 3, \cdots, N$ are constant matrices, and such that

$$[B_i(w), A(t)]=0, \quad i=0, 1, 2, \cdots \tag{28}$$

It is easy to prove that, with (27), (28) and theorem 4, the system (1) is a G.F. system.

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