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# MINIMAL P-SPACES

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**Abstract**: Minimal $s$-Urysohn and minimal $s$-regular spaces are studied. An $s$-Urysohn (respectively, $s$-regular) space $(X,\mathscr{T})$ is said to be minimal $s$-Urysohn (respectively, minimal $s$-regular) if for no topology $\mathscr{T}'$ on $X$ which is strictly weaker than $\mathscr{T}$, $(X,\mathscr{T}')$ is $s$-Urysohn (respectively $s$-regular). Several characterizations and other related properties of these classes of spaces have been obtained.

The present paper is a study of minimal $P$-spaces where $P$ refers to the property of being an $s$-Urysohn space or an $s$-regular space. A $P$-space $(X,\mathscr{T})$ is said to be minimal $P$ if for no topology $\mathscr{T}'$ on $X$ such that $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$, $(X,\mathscr{T}')$ has the property $P$. A space $X$ is said to be $s$-Urysohn [2] if for any two distinct points $x$ and $y$ of $X$ there exist semi-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $\text{cl}U\cap\text{cl}V=\phi$, where $\text{cl}U$ denotes the closure of $U$. A space $X$ is said to be $s$-regular [6] if for any point $x$ and a closed set $F$ not containing $x$ there exist disjoint semi-open sets $U$ and $V$ such that $x\in U$ and $F\subseteq V$. Throughout the paper the spaces are assumed to be Hausdorff.

## 1. Minimal $s$-Urysohn spaces

DEFINITIONS 1.1. A set $A\subseteq X$ is said to be *semi-open* [5] if there exists an open set $U\subseteq X$ such that $U\subseteq A\subseteq\text{cl}U$. The complement of a semi-open set is said to be *semi-closed* [3]. The *semi-closure* [3] of a set $A$ is the intersection of all semi-closed sets containing $A$.

DEFINITION 1.2. An $s$-Urysohn space $(X,\mathscr{T})$ is said to be *minimal s-Urysohn* if for no topology $\mathscr{T}'$ on $X$ such that $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$, $(X,\mathscr{T}')$ is $s$-Urysohn.

DEFINITION 1.3. A filter base $\mathscr{F}$ is said to be an *s-Urysohn filter base* if whenever $x$ is not an adherent point of $\mathscr{F}$, there exists a semi-open set $U$ containing $x$ such that $\text{cl}U\cap\text{cl}F=\phi$ for some $F\in\mathscr{F}$.

THEOREM 1.4. *An s-Urysohn space* $(X, \mathscr{T})$ *is minimal s-Urysohn if and only if every open s-Urysohn filter base with unique adherent point converges.*

PROOF. Let $(X, \mathscr{T})$ be minimal $s$-Urysohn. Let $\mathscr{B}$ be an open $s$-Urysohn filter base with unique adherent point $p$ to which it does not converge. Let $\mathscr{U}(x)$ be the family of all open sets containing $x$. Let $\mathscr{U}'(x)$ be the family defined as:

$$\mathscr{U}'(x)=\begin{cases} U \text{ where } U\in\mathscr{U}(x) \text{ and } x\neq p \\ U\cup B \text{ where } U\in\mathscr{U}(x),\ B\in\mathscr{B} \text{ and } x=p \end{cases}$$

Let $\mathscr{T}'$ be the topology generated by the neighbourhood base $\mathscr{U}'(x)$. Since $\mathscr{B}$ does not converge to $p$, $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$. We shall prove that $(X, \mathscr{T}')$ is $s$-Urysohn. For two distinct points $x$ and $y$ other than $p$, there exist disjoint semi-open sets $U$ and $V$ containing $x$ and $y$ respectively such that $\mathscr{T}'$-cl$U\cap\mathscr{T}'$-cl$V=\phi$. Now suppose that one of the points coincides with $p$. Let $y=p$. Since $x$ is not an adherent point of the filter base $\mathscr{B}$, There exists an open set $V$ containing $x$ such that $V\cap B=\phi$ for some $B\in\mathscr{B}$. Since $B$ is open, $\mathscr{T}$-cl$V\cap B=\phi$. Also, $(X, \mathscr{T})$ being Hausdorff, there exist open sets $V_1$ and $U$ containing $x$ and $y$ respectively such that $V_1\cap U=\phi$. Again, $\mathscr{T}$-cl$V_1\cap U=\phi$. Now $V\cap V_1$ is an open set containing $x$ and $\mathscr{T}$-cl$(V\cap V_1)\cap(B\cup U)=\phi$. $B\cup U$ being a $\mathscr{T}'$-open set, $\mathscr{T}'$-cl$(V\cap V_1)\cap(B\cup U)=\phi$. But $\mathscr{T}'$-cl$(V\cap V_1)$ is $\mathscr{T}'$-open. Therefore $\mathscr{T}'$-cl$(V\cap V_1)\cap\mathscr{T}'$-cl$(B\cup U)=\phi$ and $y\in B\cup U$. Thus $(X, \mathscr{T}')$ is $s$-Urysohn. In other words, $(X, \mathscr{T})$ is not minimal $s$-Urysohn. This is a contradiction. Therefore the open $s$-Urysohn filter base $\mathscr{B}$ converges to its unique adherent point $p$.

Conversely, let $(X, \mathscr{T})$ be an $s$-Urysohn space satisfying the condition that every open $s$-Urysohn filter base with unique adherent point converges. If possible let $\mathscr{T}'$ be an $s$-Urysohn topology on $X$ which is weaker than $\mathscr{T}$. Let $\mathscr{U}'(x)$ be the family of open sets containing $x$ in $(X, \mathscr{T}')$. $\mathscr{T}'$ being Hausdorff, $\mathscr{U}'(x)$ is an $s$-Urysohn filter base on $(X, \mathscr{T}')$ with unique adherent point $x$. $\mathscr{T}'$ being weaker than $\mathscr{T}$, $\mathscr{U}'(x)$ is an open $s$-Urysohn filter base on $(X, \mathscr{T})$ with unique adherent point and hence in view of the assumption, $\mathscr{U}'(x)$ converges to $x$ in $(X, \mathscr{T})$. Therefore each $\mathscr{T}$-neighbourhood of $x$ is a $\mathscr{T}'$ neighbourhood. Thus $\mathscr{T}=\mathscr{T}'$ and so $(X, \mathscr{T})$ is minimal $s$-Urysohn.

THEOREM 1.5. *Let* $(X, \mathscr{T})$ *be an s-Urysohn space such that every open s-Urysohn filter base with unique adherent point converges. Then every open s-*

*Urysohn filter base has non-empty adherence.*

PROOF. Let $(X, \mathscr{T})$ be an $s$-Urysohn space such that every open $s$-Urysohn filter base with unique adherent point converges. If possible, let $\mathscr{B}$ be an open $s$-Urysohn filter base without any adherent point. Let $p \in X$. Let $\mathscr{U}$ be the family of all open sets containing $p$. Let $\mathscr{C} = \{B \cup U$ where $B \in \mathscr{B}$ and $U \in \mathscr{U}\}$. Since $X$ is Hausdorff and $\mathscr{B}$ does not have an adherent point, $\mathscr{C}$ is an $s$-Urysohn filter base with unique adherent point $p$. But it does not converge to $p$. This is a contradiction. Hence the proof.

DEFINITION 1.6 [1]. An $s$-Urysohn space is said to be *s-Urysohn-closed* if it is closed in every $s$-Urysohn space in which it can be embedded.

COROLLARY 1.7. *A minimal s-Urysohn space is s-Urysohn-closed.*

PROOF. Immediate, in view of Theorems 1.4 and 1.5 above and Theorem 1.4 of [1].

THEOREM 1.8. *Every clopen subset of a minimal s-Urysohn space is minimal s-Urysohn.*

PROOF. Let $X$ be minimal $s$-Urysohn and $Y$ be a clopen subset of $X$. $Y$ being open, is $s$-Urysohn [2]. Let $\mathscr{B}$ be an open $s$-Urysohn filter base on $Y$. If possible let $\mathscr{B}$ have a unique adherent point $p \in Y$ to which it does not converge. Now $\mathscr{B}$ is an $s$-Urysohn filter base on $X$. For, suppose $x$ is not an adherent point of $\mathscr{B}$ in $X$. Suppose that every semi-open subset $V$ of $X$ containing $x$ is such that $\mathrm{cl}V \cap \mathrm{cl}B \neq \phi$ for every $B \in \mathscr{B}$. $V \cap Y$ is a semi-open subset of $X$ [3] and hence of $Y$ [5]. Since $Y$ is a semi-closed subset of $X$, $x \in Y$. Thus $Y \cap V$ is a semi-open subset of $Y$ containing $x$ and $\mathscr{T}_Y\text{-cl}(V \cap Y) \cap \mathscr{T}_Y\text{-cl}B \neq \phi$ for every $B \in \mathscr{B}$. Also $Y$ being open, every semi-open subset of $Y$ is of the form $V \cap Y$ where $V$ is a semi-open subset of $X$ [4]. Hence every semi-open subset $U$ of $Y$ containing $x$ is such that $\mathscr{T}_Y\text{-cl}U \cap \mathscr{T}_Y\text{-cl}B \neq \phi$ for every $B \in \mathscr{B}$. This is a contradiction to the fact that $\mathscr{B}$ is an $s$-Urysohn filter base on $Y$. Also $Y$ being clopen, $p$ is the unique adherent point of $\mathscr{B}$ in $X$. So in view of the given condition, $\mathscr{B}$ converges to $p$ in $X$ and hence in $Y$. Thus $Y$ is a minimal $s$-Urysohn.

THEOREM 1.9. *If* $X = \prod_{\lambda \in \Lambda} X_\lambda$ *and if there does not exist an s-Urysohn filter base on X with unique adherent point, then for at least one* $\lambda$, $X_\lambda$ *does not have an s-Urysohn filter base with unique adherent point.*

PROOF. Suppose for each $\lambda\in\Lambda$ there exists an $s$-Urysohn filter base $\mathscr{F}_\lambda$ on $X_\lambda$ with unique adherent point $x_\lambda\in X_\lambda$. We now claim that $\prod_{\lambda\in\Lambda}\mathscr{F}_\lambda$ is an $s$-Urysohn filter base on $X$ with unique adherent point $(x_\lambda)\in X$. To prove that $\prod_{\lambda\in\Lambda}\mathscr{F}_\lambda$ is an $s$-Urysohn filter base, suppose $y=(y_\lambda)$ is not an adherent point of $\prod_{\lambda\in\Lambda}\mathscr{F}_\lambda$. Then $y_\lambda$ is not an adherent point of $\mathscr{F}_\lambda$ for some $\lambda\in\Lambda$. Hence there exists a semi-open set $U_\lambda$ containing $y_\lambda$ such that $\mathrm{cl}U_\lambda\cap\mathrm{cl}F_\lambda=\phi$ for some $F_\lambda\in\mathscr{F}_\lambda$. This implies that $P_\lambda^{-1}(\mathrm{cl}U^\lambda)\cap P_\lambda^{-1}(\mathrm{cl}F_\lambda)=\phi$. Since $P_\lambda$ is continuous, $\mathrm{cl}P_\lambda^{-1}(U_\lambda)\cap\mathrm{cl}P_\lambda^{-1}(F_\lambda)=\phi$. This proves that $\prod_{\lambda\in\Lambda}\mathscr{F}_\lambda$ is an $s$-Urysohn filter base on $X$ since every semi-open subset $U$ of $X$ is of the form $\prod_{\lambda\in\Lambda}V_\lambda$ where $V_\lambda=X_\lambda$ for all but finitely many $\lambda$'s and $V_\lambda$ is a semi-open subset of $X_\lambda$ for finitely many $\lambda$'s [7]. It is easy to verify that $x=(x_\lambda)$ is the unique adherent point of $\prod_{\lambda\in\Lambda}\mathscr{F}_\lambda$.

COROLLARY 1.10. *If $X=\prod_{\lambda\in\Lambda}X_\lambda$ is minimal s-Urysohn vacously, then for at least one $\lambda$, $X_\lambda$ is minimal s-Urysohn vacously.*

THEOREM 1.11. *Let every open s-Urysohn filter base on $X\times Y$ with unique adherent point converge and let $Y$ be such that every open s-Urysohn filter base on $Y$ has a unique adherent point. Then every open s-Urysohn filter base on $X$ with unique adherent point converges.*

PROOF. Let $\mathscr{F}$ be an open $s$-Urysohn filter base on $X$ with unique adherent point $x$. Let $\mathscr{G}$ be an open $s$-Urysohn filter base on $Y$ with unique adherent point $y$. Then $\mathscr{F}\times\mathscr{G}$ is an open $s$-Urysohn filter base on $X\times Y$ with unique adherent point $(x, y)$ in $X\times Y$. In view of the given condition, $\mathscr{F}\times\mathscr{G}$ converges to $(x, y)$. Hence $\mathscr{F}$ converges to $x$.

COROLLARY 1.12. *If $X\times Y$ is minimal s-Urysohn and $Y$ is a space such that every open s-Urysohn filter base on $y$ has unique adherent point, then $X$ is minimal s-Urysohn, provided $X$ is s-Urysohn.*

## 2. Minimal s-reqular spaces

DEFINITION 2.1. An $s$-regular space $(X,\mathscr{T})$ is said to be *minimal s-regular* if for no topology $\mathscr{T}'$ on $X$ such that $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$, $(X,\mathscr{T}')$ is $s$-regular.

DEFINITION 2.2. A *filter base* is said to be an *s-regular filter base* if it is equivalent to a semi-closed filter base.

LEMMA 2.3 [6]. *A space X is s-regular if and only if for every point x and every open set U containing x, there exists a semi-open set V containing x such that* $x \subseteq V \subseteq s\text{-cl}V \subseteq U$.

THEOREM 2.4. *An s-regular space is minimal s-regular if and only if every s-regular filter base with unique adherent point converges.*

PROOF. Let $(X, \mathscr{T})$ be an $s$-regular space which is minimal $s$-regular and $\mathscr{B}$ be an $s$-regular filter base with unique adherent point $p$ to which it does not converge. For each $x \in X$, Let $\mathscr{U}(x)$ be the family of open sets containing $x$. Let us define $\mathscr{U}'(x)$ as:

$$\mathscr{U}'(x) = \begin{cases} U \text{ where } U \in \mathscr{U}(x) \text{ if } x \neq p \\ U \cup \text{cl}B \text{ where } U \in \mathscr{U}(x) \text{ and } B \in \mathscr{B} \text{ if } x = p. \end{cases}$$

Let $\mathscr{T}'$ be the topology generated by the neighbourhood base $\mathscr{U}'(x)$. Since $\mathscr{B}$ deos not converge to $p$, $\mathscr{T}'$ is strictly weaker than $\mathscr{T}$. We shall prove that $(X, \mathscr{T}')$ is $s$-regular. Let $x \in X$ and $A$ be a $\mathscr{T}'$-open set containing $x$.

Case 1 Suppose $x \neq p$. Since $(X, \mathscr{T}')$ is Hausdorff, there exists a $\mathscr{T}'$-open set $U_1$ containing $x$ and a $\mathscr{T}'$-open set $U_2$ containing $p$ such that $U_1 \cap U_2 = \phi$ and $U_1 \subseteq A$. Also since $(X, \mathscr{T})$ is $s$-regular, there exists a $\mathscr{T}$-semi-open set $V$ containing $x$ (and hence a $\mathscr{T}'$-semi-open set $V$ containing $x$) such that $x \in V \subseteq \mathscr{T}\text{-}s\text{-cl}V \subseteq U_1$. Therefore $\mathscr{T}\text{-}s\text{-cl}V \cap U_2 = \phi$. In other words, there exists a $\mathscr{T}'$-open set containing $p$ having empty intersection with $\mathscr{T}\text{-}s\text{-cl}V$. Thus $\mathscr{T}$-$s$-cl$V$ is $\mathscr{T}'$-semi-closed. Hence there exists a $\mathscr{T}'$-semi-open set $V$ such that $x \in V \subseteq \mathscr{T}'\text{-}s\text{-cl}V \subseteq U_1 \subseteq A$.

Case 11 Let $x = p$. Hence there exists a $B \in \mathscr{B}$ and a $U \in \mathscr{U}(p)$ such that $p \in U \cup \text{cl}B \subseteq A$. Since $U$ is a $\mathscr{T}$-open set containing $p$ there exists a $\mathscr{T}$-semi-open set $V$ such that $p \in V \subseteq \mathscr{T}\text{-scl}V \subseteq U$. Now there exists a $\mathscr{T}$-open set $G$ such that $G \subseteq V \subseteq \mathscr{T}\text{-cl}G \subseteq \mathscr{T}'\text{-cl}G$, since $V$ is $\mathscr{T}$-semi-open. If $p \notin G$, this proves that $V$ is a $\mathscr{T}'$-semi-open set. If $p \in G$, then $G \cup \text{cl}B \subseteq V \cup \text{cl}B \subseteq \text{cl}G \cup \text{cl}B \subseteq \mathscr{T}\text{-cl}(G \cup \text{cl}B) \subseteq \mathscr{T}'\text{-cl}(G \cup \text{cl}B)$. Since $G \cup \text{cl}B$ is a $\mathscr{T}'$-open set, this implies that $V \cup \text{cl}B$ is $\mathscr{T}'$-semi-open. Thus $p \in V \cup \text{cl}B \subseteq s\text{-cl}V \cup \text{cl}B \subseteq U \cup \text{cl}B \subseteq A$. We shall prove that $s\text{-cl}V \cup \text{cl}B$ is $\mathscr{T}'$-semi-closed. Suppose that $x \notin (\text{scl}V \cup \text{cl}B)$. Then $x \neq p$. Therfore there exists a $\mathscr{T}$-semi-open set (and hence a $\mathscr{T}'$-semi-open set) $H$ containing $x$ such that $H \cap (s\text{-cl}V \cup \text{cl}B) = \phi$ since $(s\text{-cl}V \cup \text{cl}B)$ is $\mathscr{T}$-semi-closed. Hence $(s\text{-cl}V \cup \text{cl}B)$ is $\mathscr{T}'$-semi-closed. Thus $p \in V \cup \text{cl}B \subseteq \mathscr{T}'\text{-}s\text{-cl}(V \cup \text{cl}B) \subseteq A$. Hence $(X, \mathscr{T}')$ is $s$-regular. This contradicts the fact that $(X, \mathscr{T})$ is minimal $s$-regular. Therefore $\mathscr{B}$ converges to $p$.

The converse can be proved as in the proof of Theorem 1.4.

THEOREM 2.5. *Let $(X, \mathscr{T})$ be an s-regular space such that every s-regular filter base with unique adherent point converges. Then every s-regular filter base on X has non-empty adherence.*

PROOF. Similar to the proof of Theorem 1.5

DEFINITION 2.6 [1]. An $s$-regular space $(X, \mathscr{T})$ is said to be *s-regular-closed* if it is closed in every $s$-regular space in which it can be embedded.

COROLLARY 2.7. *A minimal s-regular space is s-regular-closed.*

PROOF. Immediate, in view of Theorems 2.4 and 2.5 above and Theorem 2.4 of [1].

THEOREM. 2.8. *If a subset Y of s-regular space X has the property that every s-regular filter base on Y has non-empty adherence, then Y is a closed subset of X.*

PROOF. Suppose $Y$ is not a closed subset of $X$. Let $p \in \text{cl}y - y$. Let $\mathscr{U}$ and $\mathscr{V}$ be filter bases consisting of open subsets of $X$ containing $p$ and semi-closures of semi-open subsets of $X$ containing $p$ respectively. Let $\mathscr{B} = \{Y \cap U : U \in \mathscr{U}\}$ and $\mathscr{C} = \{y \cap V : V \in \mathscr{V}\}$. $\mathscr{B}$ and $\mathscr{C}$ are filter bases on $Y$ where $\mathscr{C}$ is a semi-closed filter base. To see that $\mathscr{C}$ is a family of semi-closed subsets of $Y$, let $V \in \mathscr{V}$. Then there exists a closed subset $W$ of $X$ such that $\text{int}W \subseteq V \subseteq W$ [3]. Hence $Y \cap \text{Int}W \subseteq Y \cap V \subseteq y \cap W$. Thus $\text{int}(y \cap W) \subseteq y \cap V \subseteq y \cap W$ where $y \cap W$ is a closed subset of $y$. This proves that $y \cap V$ is a semi-closed subset of $Y$. Since $X$ is $s$-regular, $\mathscr{U}$ and $\mathscr{V}$ are equivalent in $X$ and so $\mathscr{B}$ and $\mathscr{C}$ are equivalent in $Y$. In other words $\mathscr{B}$ is an $s$-regular filter base on $y$ with unique adherent point $p$ and $p \notin Y$. This is a contradiction. Hence $y$ is a closed subset of $X$.

THEOREM 2.9. *Every clopen subset of a minimal s-regular space is minimal s-regular.*

PROOF. Let $X$ be minimal $s$-regular and let $Y$ be a clopen subset of $X$. Let $\mathscr{B}$ be an $s$-regular filter base on $Y$. If possible let $\mathscr{B}$ have a unique adherent point $p$ in $Y$ to which it does not converge. Since $Y$ is clopen, $\mathscr{B}$ is an $s$-regular filter base on $X$ with unique adherent point $p$. Hence $\mathscr{B}$ converges to $p$. But $p \in Y$. So converges in $Y$. Hence $(Y, \mathscr{T}_Y)$ is minimal $s$-regular.

THEOREM 2.10. *If* $X=\prod_{\lambda\in\Lambda} X_\lambda$ *is minimal s-regular then each* $X_\lambda$ *is minimal s-regular provided each* $X_\lambda$ *is s-regular.*

PROOF. Let $X=\prod_{\lambda\in\Lambda} X_\lambda$, with the product topology $\mathscr{T}$, be minimal $s$-regular. Suppose $(X_{\lambda_0}, \mathscr{T}_{\lambda_0})$ is not minimal $s$-regular for some $\lambda_0\in\Lambda$. Then there exists an $s$-regular topology $\mathscr{F}'_{\lambda_0}$ on $X_{\lambda_0}$ strictly weaker than $\mathscr{F}_{\lambda_0}$. Consider now the collection $\{(X_\beta, \mathscr{T}_\beta) : (X_\beta, \mathscr{T}_\beta)=(X_\lambda, \mathscr{T}_\lambda)$ for $\lambda\neq\lambda_0$ and $(X_\beta, \mathscr{T}_\beta)=(X_\lambda, \mathscr{F}'_{\lambda_0})$ if $\lambda=\lambda_0$. Then $X=\prod_{\beta\in\Lambda} X_\lambda$ has the product topology $\mathscr{T}'$ which is strictly weaker than $\mathscr{T}$. Also $(X, \mathscr{T}')$ is $s$-regular since each $X_\lambda$ is $s$-regular [8]. Thus each $(X_\lambda, \mathscr{T}_\lambda)$ is minimal $s$-regular.

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