Kyungpook Math. J.
Volume 27, Number 1
June, 1987.

OSCILLATORY BEHAVIOUR OF SOLUTIONS OF $y''+P(x)y=f(x)$

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Abstract: This paper is a study of the oscillatory and asymptotic behaviour of solutions of the second order nonhomogeneous linear differential equation $y''+P(x)y=f(x)$, and the associated homogeneous equation. Conditions are determined, under which the nonhomogeneous equation is oscillatory if and only if the homogeneous equation is oscillatory.

1. Introduction

This paper is concerned with the oscillatory and asymptotic behaviour of solutions of the second order nonhomogeneous linear differential equation

$$y''+P(x)y=f(x), \qquad (NH)$$

in relation to the associated homogeneous equation

$$y''+P(x)y=0, \qquad (H)$$

where $P(x)$ and $f(x)$ are assumed to be continuous real-valued functions on the infinite half axis $[a, \infty)$, for some real number a. It is also assumed that $f'(x)\leq 0$, $p'(x)\geq 0$. The authors [1-8], as example, have obtained results for these equations. This paper will extend their work.

DEFINITIONS. (1) We say a solution of (NH) is *oscillatory* on $I=[a, \infty)$ if it has on infinite number of zeros on $[\alpha, \infty)$ for every $\alpha\geq 0$. A solution is said to be *nonoscillatory* on I if it has only a finite number of zeros on I for some $x\geq a$. Further, the equation (NH) is *oscillatory* if it has at least one oscillatory solution and is nonoscillatory if all solutions are nonoscillatory.

(2) A solution of (H) is oscillatory if it has an infinite number of zeros on I and is *nonoscillatory* if it has only a finite number of zeros on I. As for (NH), the equation (H) is oscillatory if it has at least one oscillatory solution and is nonoscillatory if no solution of (H) is oscillatory on I. By the Strum interlacing theorem, (H) is oscillatory if and only if all solutions are oscillatory.

(3) A solution of (H) is called *quickly oscillatory* if it is oscillatory and sequence of zeros $\{x_n\}$ is such that $\lim_{n\to\infty}(x_{n+1}-x_n)=0$.

The difference between an oscillatory solution of (H) and an oscillatory solution of (NH) is illustrated by the simple equation $y''(x)=f(x)$, where

$$f(x)=\begin{cases} 0 & , \ x=0 \\ (20x^3-x)\sin\frac{1}{x}-8x^2\cos\frac{1}{x}, & x\in(0,\ \infty). \end{cases}$$

It is easily verified that $y(x)=x^5\sin\left(\frac{1}{x}\right)$ is a solution of this equation. Moreover, this solution has infinitely many zeros on $[0,\ 1]$, and yet this solution is nonzero on $[1,\ \infty)$. Such a solution, we wish to call nonoscillatory. This behaviour does not occur in equation (H) since the only solution of (H) having infinitely many zeros on a finite interval is the trivial solution (i. e. the identically zero solution). However, one can show that if $f(x)\neq 0$ on any finite interval, then any solution of (NH) has only a finite number of zeros on that intervals.

Let $u(x)$ and $v(x)$ be any two functions of C^1-class, the Wronskian $W[u, v](x) =u(x)v'(x)-u'(x)v(x)$. If $\{u(x), v(x)\}$ is a solution basis for (H) on I, it follows that $W[u, v](x)=k$, where k is nonzero constant. The solution basis will be called a ***normalized solution*** basis for (H) if $k=1$.

Throughout this paper, we will use frequently the following lemma (Burton [2], Leighton [5]).

LEMMA A. *If $W[u, v](x)\neq 0$ on $[a,\ \infty)$, then the zeros of $u(x)$ and $v(x)$ separate each other on I.*

In our analysis we shall frequently employ the following identity which may be verified by differentiation,

$$W[u,\ y](x)=\int_a^x f(t)u(t)dt, \tag{1.1}$$

where $y(x)$ is the solution of (NH) and $u(x)$ is any solution of (H). The particular solution y_p of (NH) can be written in the form

$$y_p=\int_0^x \begin{vmatrix} u(x) & v(x) \\ u(t) & v(t) \end{vmatrix} f(t)dt,$$

where $\{u,\ v\}$ is normalized solution basis for (H), and hence the solution y of (NH) has the form

$$y(x)= [C_1-\int_a^x f(t)v(t)dt]\, u(x)+ [C_2+\int_a^x f(t)u(t)dt]\, v(x),$$

where C_1 and C_2 are arbitrary constants.

2. Main results

Now, we are able to study the oscillatory and asymptotic behaviour of solutions of (NH) and (H).

THEOREM 1. *The function* $W[u,\ y](x)\neq 0$ *on* I *if and only if* $y(x)$ *has only simple zeros and the zeros of* $y(x)$ *and* $u(x)$ *separate each other on* I.

PROOF. Lemma A implies that if $W[u,\ y](x)\neq 0$ on I, then the zeros of $y(x)$ are simple and the zeros of $y(x)$ and $u(x)$ separate each other on I independent of $f(x)$. Now, assume that x_1 and x_2 are any two consecutive zeros of $y(x)$, x_3 and x_4 are two consecutive zeros of $u(x)$, such that, $x_1<x_3<x_2<x_4$.
We can assume, without loss of generality $y(x)>0$ on $(x_1,\ x_2)$ and $u(x)<0$ on $(x_3,\ x_4)$. It follows that $W[u(x_i),\ y(x_i)]>0$, $i=1,2,3,4$. We have, from (1.1), $W[u,\ y](x)=\int_a^x f(t)u(t)dt$. If there exists a point $\xi\in(x_1,\ x_3)$ such that $W[u,\ y](\xi)=0$, then there must exist a point $\eta\in(x_1,\ x_3)$ such that $W'[u,y](\eta)=0$. This implies $f(\eta)u(\eta)=0$ or $u(\eta)=0$ contradicting the separation of zeros of $u(x)$ and $y(x)$. Then, it follows that $W[u,y]\neq 0$ on $(x_1,\ x_3)$ and similarly $W[u,\ y]\neq 0$ on $(x_3,\ x_4)$. This completes the proof of the theorem.

THEOREM 2. *If* $f(x)$ *is a solution of* (H), *then* (NH) *is oscillatory if and only if* (H) *is oscillatory.*

PROOF. From (1.1) we obtain $W[f,\ y](x)=\int_a^x f^2(t)dt$. It follows that $W'[f,\ y]\geq 0$. Since $f(x)\neq 0$ on any subinterval of I, then $W[f,\ y]$ has at most one zero on $[a,\ \infty)$ and the result follows.

THEOREM 3. *If* (H) *is oscillatory, then all nonoscillatory solutions of* (NH) *are eventually of the same sign. Moreover, if* $f(x)\neq 0$ *for large* x, *then all nonoscillatory solutions of* (NH) *are eventually of the same sign as* $f(x)$.

PROOF. Let (H) be oscillatory, and suppose that y_1 and y_2 are nonoscillatory solutions of (NH) such that the sign $y_1(x)\neq$sign $y_2(x)$ for large x. Then, y_1-y_2 is nonoscillatory solution of (H) which is a contradiction to the hypothesis. Suppose that $f(x)\neq 0$ on I and that $y(x)$ is nonoscillatory solution of (NH) such that sign $y(x)\neq$sign $f(x)$ on $[b,\ \infty)$, $b\geq a$. It follows that y is a solution of the homogeneous equation $\omega''+(p-f/y)\omega=0$.
By Strum comparison theorem, this implies (H) must be nonoscillatory, which is a contradiction. Hence, sign $y=$ sign f for large x. This completes the proof.

Equation (NH) can be written in the form:

$$y''=p(x)\ [h(x)-y(x)]$$

where $h(x)=f(x)/p(x)$, $p(x)\neq 0$ for every $x\in[a,\infty)$, $h(x)\in C^2$. We note that $y(x)$ is a solution of (NH) then $\omega(x)=y(x)-h(x)$ is a solution of $\omega''+p(x)\omega=-h''(x)$. If an integration by parts is performed on the left hand member of (1.1) we get

$$\int_a^x h'(t)u'(t)dt=[y'(t)u(t)-(y(t)-h(t))u'(t)]_a^x. \qquad (2.1)$$

THEOREM 4. *Suppose (H) is oscillatory and $h(x)\in C^2$*

(i) *If $h(t)$ is an increasing function and bounded above, then no solution stays above $h(x)$ on I.*

(ii) *If $h(x)$ is a decreasing function and bounded below, then no solution stays above $h(x)$ on I.*

PROOF. We shall prove (ii), the proof of (i) straightforward. Let $h(x)$ be the solution of (H) such that $u(a)>0$, $u'(a)=0$. It follows that there exists a point $b>a$ such that $u'(b)=0$ and $u'(x)<0$ on (a, b). If $y(x)$ is a solution of (NH) with $y(a)>h(a)$, then either $y'(a)\leq 0$ or by using identity (2.1) with $x=b$, $y'(b)<0$. In either case, since $y'(x)<0$ for $y(x)>h(x)$, it follows that $y(x)$ must cut $h(x)$. This completes the proof.

THEOREM 5. *Let $f(x)$ change sign on every interval (b,∞), $b\geq a$. If between every pair of succesive sign change points b_1, b_2, (with $(b_1<b_2)$), of $f(x)$, there exists a nontrivial solution of (H) such that $u(b_1)=0$ and $u(c)=0$ for $C\in[b_1, b_2]$, then (NH) is oscillatory.*

PROOF. Suppose that $y(x)$ is nonoscillatory. Consider the cases:

(i) Assume $y(x)>0$ for $x\geq d$, $d>a$, and let b_1 and b_2 be consecutive sign change points of $f(x)$ with $b_2>b_1>d$. Rewrite (1.1) in the form

$$W[y, u]=\int_{b_1}^c f(t)u(t)dt. \qquad (2.2)$$

Our hypothesis allows us to assume that $f(x)\leq 0$ in $[b_1, b_2]$ and $u>0$ in (b_1, c). It follows, from (2.2) that $y(c)u'(c)>y(b_1)u'(b_1)>0$, which contradicts that fact that $y(c)u'(c)<0$.

(ii) Assume that $y(x)<0$ for $x\geq d$, $d>a$, then we can find consecutive sign change points b_1 and b_2 of $f(x)$ such that $f(x)\geq 0$ on $[b_1, b_2]$ and a solution $u(t)$ of (H) with $u(b_1)=u(c)=0$ and $u>0$ on (b, c) for some $c\in[b_1, b_2]\subset I$. Again by (2.2) we arrive at a contradiction. This completes the proof.

THEOREM 6. *Suppose that (H) is oscillatory and let the distance between consecutive sign change points of* $f(x)$, b_1, b_2 *with* $b_2>b_1$ *be bounded below by a constant* $M>0$. *If for any solution* $u(x)$ *of (H) every pair consecutive zeros* x_1, x_2 *are such that* $|x_1-x_2|\leq M$, *then (NH) is oscillatory.*

PROOF. Assume that $u(b_1)=0$ and $u'(b_1)>0$. Our hypothesis implies $u(x)$ to have another zero in $(b_1, b_2]$. By applying arguments used to prove theorem 5, and then result follows.

THEOREM 7. *If (H) is quick oscillatory and* $f(x)$ *change sign on* (b, ∞) *for each* $b\geq a$ *and if the distance between consecutive sign change points bounded below, then (NH) is oscillatory.*

PROOF. Since (H) is quickly oscillatory and the distance between consecutive zeros of (H) is eventually less than the lower bound of the consecutive sign change points of $f(x)$. By applying theorem 6, the result follows.

THEOREM 8. *If* $\lim_{x\to\infty} P(x)=\infty$ *and* $f(x)$ *changes sign on* (b, ∞) *for each* $b>a$ *such that the distance between consecutive sign change points is bounded below, then (H) is oscillatory.*

PROOF. The hypothesis $\lim_{x\to\infty} P(x)=\infty$ implies (H) is quickly oscillatory and the results follow from theorem 7.

THEOREM 9. *If (H) is oscillatory, then every pair of nonosillatory solutions of (NH) eventually has the same sign.*

PROOF. Suppose $y_1(x)$ and $y_2(x)$ are nonoscillatory solutions of (NH). It follows $u(x)=y_1(x)-y_2(x)$ is a solution of (H). The assumption that (H) is oscillatory implies that (H) is oscillatory implies that there exists an infinite sequence $\{x_n\}$ of zeros of $u(x)$. It follows that $y_1(x_n)=y_2(x_n)$ for all n. Thus, $y_1(x)$ and $y_2(x)$ have the same sign for large x.

THEOREM 10. *If* $\int_a^{\infty} p(x)dx=\infty$, *then every pair of nonoscillatory solutions on (NH) eventually have the same sign.*

PROOF. The hypothesis implies (H) is oscillatory. The result follows by applying theorem 9.

To derive further properties for (NH) we require the following theorem of Keener [6].

THEOREM A. *If $f(x)>0$, $f'(x)\geq 0$, $p'(x)\leq 0$ for $x\in[a, b]$ and if $f(x)$ and $p(x)$ are not constant on a common subinterval, then a solution y of (NH) such that $y(a)=y'(a)=0$ is positive on interval $(a, b]$.*

This theorem enables us to compare the solution of

$$y''+p(x)y=f_1(x), \tag{2.3}$$

$$z''+p(x)z=f_2(x), \tag{2.4}$$

where $f_1(x)$, $f_2(x)$ and $p(x)$ are of C^1-class on $[a, b]$.

THEOREM 11. *Let $f_1>f_2$, $f_1'\geq f_2'$, $p'\leq 0$ on $[a, b]$. Assume also $f_3=f_1-f_2$ and $p(x)$ are not constants on common subinterval. If $y(x)$ and $z(x)$ are solutions of (2.3) and (2.4) respectively and satisfies $y(a)=z(a)=y'(a)=z'(a)=0$, then $y(x)>z(x)$ for $x\in(a, b]$.*

PROOF. Subtracting (2.4) from (2.3) we have

$$\omega''+p(x)\omega=f_3(x),$$

where $\omega=y-x$. Since $f_3(x)>0$, $f'(x)\geq 0$ and since ω has a double zero at $x=a$. Applying theorem A it follows that $\omega(x)>0$ for $(a, b]$.

We remark that if $f_3(x)$ and $p(x)$ are not constants on common subinterval, the conclusion of theorem 11 follows i.e. $y(x)>z(x)$, $x\in(a, b]$. This completes the proof.

As example consider the equations

$$y''+y=1+e^{-x} \text{ and } z''+z=e^{-x},$$

with $a=0$. The solutions are

$$y=1+\frac{1}{2}(e^{-x}+\sin x-3\cos x)$$

and

$$z=\frac{1}{2}(e^{-x}+\sin x-\cos x),$$

satisfy the conditions $y(0)=y'(0)=z(0)=z'(0)=0$ we have

$$y-z=1-\cos x\geq 0 \text{ on } [0, \infty).$$

Acknowledgements

The authors would like to express their appreciations to Prof. Hassan El-Owaidy, Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City, Cairo, for his comments on the manuscript.

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