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ON THE HYPOELLIPTIC BOUNDARY VALUE PROBLEMS

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1. Introduction

Let $P(D, D_t)$ be a hypoelliptic differential operator of type μ with constant coefficients. Let Ω be an open subset of the half space $\boldsymbol{R}_+^{n+1}$ with plane piece of boundary ω contained in $\boldsymbol{R}_0^n$. Let $Q_1(D, D_t), \cdots, Q_\mu(D, D_t)$ be μ partial differential operators with constant coefficients and consider the boundary value problem:

$$P(D, D_t)u=f \text{ in } \Omega \tag{1}$$
$$Q_\nu(D, D_t)u|_\omega=g_\nu,\ 1\leq\nu\leq\mu$$

Hörmander [4] gave a necessary and sufficient condition, based on the variety of zeros of the characteristic function of the boundary problem, in order that all solutions of (1) shall belong to C^∞ whenever the initial data belong to such class (called hypoelliptic boundary value problem). In this paper we give another characterization of this problem.

For completeness we briefly introduce the hypoelliptic differential operators. Differential operator $P(D)$ is called hypoelliptic if, for every open set Ω in $\boldsymbol{R}^{n+1}$ and every distribution u in Ω, $P(D)u\in C^\infty(\Omega)$ implies $u\in C^\infty(\Omega)$. Particularly, algebraic characterizations of hypoelliptic differential operator with constant coefficients are given by Hörmander [5] and Björck [1] as follows:

(2) $|\mathrm{Im}\ \zeta|\to\infty$ when $|\zeta|\to\infty$ on the surface $P(\zeta)=0$.

(3) for each $A>0$ there is a constant B such that $|\mathrm{Im}\ \zeta|\geq A\log(1+|\mathrm{Re}\ \zeta|)-B$ on the surface $P(\zeta)=0$.

In all that follows we assume that $P(D, D_t)$ is hypoelliptic differential operator with constant coefficients. We shall consider the root of the equation

$$P(\xi, \tau)=0. \tag{4}$$

where $\xi\in\boldsymbol{R}^n$. If τ is a real root, it follows from (2) that ξ belongs to the compact set in $\boldsymbol{R}^n$ defined by $P(\xi, \tau)=0$. That is, if ξ is outside a compact set K (we take a sphere) in $\boldsymbol{R}^n$, (4) has no real root. Since the roots are continuous function of ξ ([3], p239), in each component of the complement of

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K the number of roots with positive imaginary part is constant. We shall define that $P(D, D_t)$ is of determined type μ if the number of zeros with positive imaginary part is μ for all ξ in the complement of K. When $n>1$ all hypoelliptic differential operator with constant coefficients are thus of determined type.

2. Hypoelliptic boundary problems

Let $P(D, D_t)$ be hypoelliptic differential operator of degree k and of determined type μ. We shall denote by $\mathscr{A}$ the set of all $\zeta \in C^n$ such that the equation.

$$P(\zeta, \tau)=0 \tag{5}$$

has exactly μ roots with positive imaginary part and none of that is real. Obviously $\mathscr{A}$ is open in C^n and by hypothesis a real ξ is in $\mathscr{A}$ if ξ belongs to a suitable neighborhood of infinity. We shall estimate the size of $\mathscr{A}$ more precisely.

LEMMA 1. *Suppose that $P(D, D_t)$ is hypoelliptic and of determined type μ. Then, given any number $A>0$, there is a number B such that $\mathscr{A}$ contains all ζ satisfying*

$$|\mathrm{Im}\ \zeta| < A \log(1+|\mathrm{Re}\ \zeta|)-B. \tag{6}$$

PROOF. Taking the same B in (3), we note that if τ is real and (6) is fulfilled, we have

$$|\mathrm{Im}(\zeta, \tau)| = |\mathrm{Im}\ \zeta| < A \log(1+|\mathrm{Re}\ \zeta|)-B \leq A \log(1+|\mathrm{Re}(\zeta, \tau)|)-B$$

which implies $P(\zeta, \tau)\neq 0$ in virtue of (3), Thus (5) has no real root if (6) is valid and hence the number of roots of (5) with positive imaginary part is constant in each components of the set defined by (6). Now each components of this set contains real points with arbitrarily large absolute values which proves the lemma.

When $\zeta \in \mathscr{A}$, we denote by $\tau_1(\zeta), \cdots, \tau_\mu(\zeta)$ the zeros of (5) with positive imaginary part and set

$$K_\zeta(\tau) = \prod_{\nu=1}^{\mu} (\tau - \tau_\nu(\zeta))$$

and

$$C(\zeta) = \frac{\det(Q_l(\zeta, \tau_k(\zeta)))_{1\leq k, l\leq \mu}}{\prod_{l<k} (\tau_k(\zeta)-\tau_l(\zeta))}. \tag{7}$$

The function $C(\zeta)$ is called the characteristic function of the boundary problem

(1). $C(\zeta)$ is defined even in the case of repeated roots ([4]. p231).

Our main result is to prove the following theorem

THEOREM. *Let $P(D, D_t)$ be hypoelliptic and of determined type μ, Then the following are equivalent;*

(*a*) *The boundary value problem* (1) *is hypoelliptic.*

(*b*) $|\mathrm{Im}\ \zeta|\to\infty$ *if* $|\zeta|\to\infty$ *in* $\mathscr{A}$ *satisfying* $C(\zeta)=0$.

(*c*) *Given any number A there is a number B such that* $\zeta\in C^n$, $|\mathrm{Im}\ \zeta|\leq A$ *and* $|\mathrm{Re}\ \zeta|\geq B$ *implies* $\zeta\in A$ *and* $C(\zeta)\neq 0$.

(*d*) *Given any number* $A>0$ *there is a number B such that* $|\mathrm{Im}\ \zeta|<A\log(1+|\mathrm{Re}\ \zeta|)-B$ *implies* $\zeta\in\mathscr{A}$ and $C(\zeta)\neq 0$ *for* $\zeta\in C^n$ *with* $|\mathrm{Re}\ \zeta|\geq 1$.

PROOF. In [4] and [1] the equivalence of (a), (b) and (c) are given. Obviously (d) implies (c). It remains to show that (a) implies (d). We may assume that Ω is bounded ([4], p250). By Ω' we shall denote a domain whose closure is contained in $\Omega\cup\omega$ but not in Ω. For the proof we need the following lemma which is a slight modification of lemma 4.1 [4]. We give a sketch of the proof just for completeness.

LEMMA 2. *Under the assumptions of the theorem, given each integer j, there is a constant C depending on j such that*

$$\sum_{|\alpha|\leq k+j}\sup_{x\in\Omega'}|D^\alpha u(x)|\leq C\sum_{|\alpha|\leq k}\sup_{x\in\Omega}|D^\alpha u(x)| \tag{8}$$

for all $u\in C^k(\Omega\cup\omega)$ *satisfying* (1) *with zero initial data, where k denote the maximum of orders of* $P(D, D_t)$ *and* $Q_\nu(D, D_t)$, $1\leq\nu\leq\mu$.

SKETCH OF THE PROOF. In lemma 4.1 [4] we denote by V the space of functions $v\in C^{k+1}(\Omega')$ with bounded derivative up to order $k+j$ and the norm defined by

$$\sum_{|\alpha|\leq k+j}\sup_{x\in\Omega'}|D^\alpha v(x)|.$$

The proof goes exactly the same in lemma 4.1 [4].

Consider the homogeneous boundary problem

$$\begin{aligned} &P(D, D_t)u=0 \text{ in } \Omega \\ &Q_\nu(D, D_t)u|_\omega=0,\ 1\leq\nu\leq\mu. \end{aligned} \tag{9}$$

Applying the inequality (8) to "exponential solutions" of the boundary problem (9), that is, solutions of the form

$$u(x, t)=e^{i\langle x, \zeta\rangle}v(t)$$

where v is a function of a real variable. From the identity $P(D, D_t)u = e^{i<x,\zeta>}P(\zeta, D_t)v(t)u$ satisties (9) if and only if

$$(10) \qquad P(\zeta, D_t)v(t)=0 \quad \text{and}$$

$$(11) \qquad (Q_\nu(\zeta, D_t)v)(0)=0, \ 1\leq\nu\leq\mu.$$

Assume $\zeta\in\mathscr{A}$ and $C(\zeta)=0$. Then we can find a nontrivial function $v(t)$ satisfying (11) and

$$(12) \qquad k_\zeta(D_t)v(t)=0,$$

which satisfies (10), since $k_\zeta(\tau)$ is a factor of $P(\zeta, \tau)$.

Differentiation of an exponential solution with respect to a boundary variable x_j, $1\leq j\leq n$, is equivalent to multiplying by ζ_j. Thus it follows from (8) that

$$(13) \qquad (\sum_{i=1}^n |\zeta_i|)^j \sum_{|\alpha|\leq k} \sup_{\Omega'}|D^\alpha u(x, t)|$$
$$\leq C \sum_{|\alpha|\leq k} \sup_{\Omega}|D^\alpha u(x,t)|$$

for the exponential solution $u(x, t)=e^{i<x,\zeta>}v(t)$. Now $D^\alpha u(x,t)=e^{i<x,\zeta>}v_\alpha(t)$, where $v_\alpha(t)$ is also a solution of (9). Denoting by H the supremum of $|x|$ when $(x, t)\in\Omega$ for some $t>0$, we have

$$e^{-H|\mathrm{Im}\,\zeta|}\leq|e^{i<x,\zeta>}|\leq e^{H|\mathrm{Im}\,\zeta|}, \ (x, t)\in\Omega.$$

Hence (13) gives

$$(14) \qquad (\sum_{i=1}^n |\zeta_i|)^j \sum_{|\alpha|\leq k} \sup_{\Omega'}|v_\alpha(t)|\leq Ce^{2H|\mathrm{Im}\,\zeta|} \sum_{|\alpha|\leq k} \sup_{\Omega}|v_\alpha(t)|.$$

Now let a be a positive number such that any t between 0 and a has an $x\in\boldsymbol{R}^n$ with $(x, t)\in\Omega'$ and let b be an upper bound of t with $(x, t)\in\Omega$ for some $x\in\boldsymbol{R}^n$.

Then it follows from (14) that

$$(15) \qquad (\sum_{i=1}^n |\zeta_i|)^j \sum_{|\alpha|\leq k} \sup_{0<t<a}|v_\alpha(t)|\leq Ce^{2H|\mathrm{Im}\,\zeta|} \sum_{|\alpha|\leq k} \sup_{0<t<b}|v_\alpha(t)|.$$

We now use the fact that all v_α are solutions of the equation (9) and that the zero of $k_\zeta(\tau)$ has non-negative imaginary part, it follows from theorem 1.4 ([4], p248) that

$$(16) \qquad \sup_{0<t<b}|v_\alpha(t)|\leq\gamma\left(\frac{b}{a}\right)^{\gamma-1}\sup_{0<t<a}|v_\alpha(t)|$$

for some γ depending on μ. Combining (20) and (21) and noting that $v_\alpha\neq0$ we get

$$(17) \qquad |\mathrm{Re}\,\zeta|^j\leq Ce^{2H|\mathrm{Im}\,\zeta|}$$

Taking logarithm in both side

$$j\log|\mathrm{Re}\,\zeta|\leq \log C+2H|\mathrm{Im}\,\zeta|,$$

Since $\log(1+|\mathrm{Re}\,\zeta|)\leq 1+\log|\mathrm{Re}\,\zeta|$ for $|\mathrm{Re}\,\zeta|\geq 1$, we have

$$j\log(1+|\mathrm{Re}\,\zeta|)\leq C+j+2H|\mathrm{Im}\,\zeta|,$$

that is,

$$|\mathrm{Im}\,\zeta|\geq \frac{j}{2H}\log(1+|\mathrm{Re}\,\zeta|)-\frac{C+j}{2H} \tag{18}$$

for $\zeta\in\mathscr{A}$ and $C(\zeta)=0$. Since j is arbitrary, (18) proves our claim.

REMARK. Condition (d) gives more precise geometric picture for the location of the zeros of the characteristic function $C(\zeta)$ of the hypoelliptic boundary value problem than so does (b) or (c).

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