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A Comparison between Abel-Fourier and Digital Linear Filter Methods

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Abstract: Hankel transform of order 0 can be represented by a combination of Abel transform and Fourier transform. This Abel-Fourier method can offer computational advantages over a conventional digital linear filter method through the uses of a rapid Abel transform with shift variant recursive filter and of a fast Fourier transform. The Abel-Fourier method, however, is generally less accurate than a well-designed digital linear filter method. In geoelectrical applications, the digital linear filter method seems to be more flexible than the Abel-Fourier method.

INTRODUCTION

The Hankel, or Fourier-Bessel, transform is an important tool for geophysics, acoustics and optics. In geoelectromagnetics, digital linear filter method (DLFM) is frequently used to evaluate the Hankel transform integral. Following the work of Ghosh (1971), many new filter coefficients were designed for problems occurring in geoelectrical prospecting methods (Koefoed, 1979; Patra and Mallick, 1980). The reason for intensive research in this area is that DLFM is approximately an order of magnitude faster than a direct integration of Hankel transform (Anderson, 1979). In fact, numerical convolutions using predetermined coefficients completely avoid Bessel function evaluations which are needed in the numerical integration.

In optics and acoustics, another method based on the projection-slice theorem of tomography is known in evaluating the Hankel transform. The method uses a formal equivalency between Hankel transform and Abel transform followed by Fourier transform (Oppenheim et al., 1978). Wilson (1986) used this Abel-Fourier method (AFM) for computing synthetic seismograms and for plane-wave decomposition of observed seismograms.

The AFM may offer computational advantages over other techniques through the use of fast Fourier transform (FFT) if an efficient algorithm for computing the Abel transform is known. One of such efficient algorithms, shift variant recursive filter, was designed by Hansen and Law (1985). Hansen (1985) developed a fast algorithm of Hankel transform using the shift variant filter and FFT.

In this paper, DLFM and AFM are compared with respect to computational efficiency and accuracy. Although the AFM may be applied to Hankel transforms of any order, only 0-order transform is investigated in this study. For the DLFM, Anderson's adaptive filter (Anderson, 1979) is used because of its accuracy and flexibility.

ABEL-FOURIER METHOD

AFM is based on the projection-slice theorem, with which Bracewell (1956) showed that Hankel transform is formally equivalent to Abel transform followed by Fourier transform. For a axisymmetric function $f(r)$, the projection is given by the Abel transform (Bracewell, 1978)

$$g(y)=\int_{|y|}^{\infty}\frac{2f(r)dr}{[1-(y/r)^2]^{1/2}}. \quad (1)$$

Calculating the Fourier transform of $g(y)$, we

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have

$$F(\rho)=\int_{-\infty}^{\infty} dy\ e^{-j2\pi\rho y} \times \int_0^{\infty} \frac{2f(r)U[1-(y/r)^2]}{[1-(y/r)^2]^{1/2}} dr, \quad (2)$$

where U indicates the unit step function. Reversing the order of integration, and letting $x=y/r$, (2) becomes

$$F(\rho)=\int_0^{\infty} dr\ 2rf(r) \int_{-1}^{1} \frac{e^{-j2\pi\rho r x}}{(1-x^2)^{1/2}} dx. \quad (3)$$

The second integral is equal to $\pi J_0(2\pi\rho r)$, where J_0 is the Bessel function of order 0. Consequently,

$$F(\rho)=2\pi \int_0^{\infty} f(r)\ J_0(2\pi\rho r)\ r\ dr. \quad (4)$$

The Abel transform (1) can be interpreted as a linear system with input $f(r)$, output $g(y)$, and an impulse response

$$h(x)=U(1-x^2)/(1-x^2)^{1/2}. \quad (5)$$

Hansen(1985) designed a fast algorithm of 9-order recursive filter for computing the Abel transform. The recursive filter takes uniformly spaced samples of the function $f(r)$ and produces uniformly spaced results of the Abel transform $g(y)$ for $y \geq 0$. The Hankel transform is completed by using FFT to compute the Fourier transform of $g(y)$.

Hansen's 9-order filter can reduce to a 8-order filter without any change of results because the first eigenvalue of the Hansen's filter is 0. In this paper, the modified 8-order filter is implemented in computing the Abel transform of a function. To perform the Fourier transform of the Abel transform, a split-radix FFT (Sorensen et al., 1986) is used in this paper. Therefore, the entire Hankel transform requires real multiplications of $O(3KN+2N\log N)$, where N is the number of data and K is the order of the shift variant recursive filter which is 8 in this implementation.

NUMERICAL EXPERIMENTS

Hansen (1985) tested his algorithm for four known transform pairs and showed that the algorithm works well. In this experiment, sampling ranges of all tested functions were [0, 1]. In problems of geoelectrical prospecting, however, the sampling range [0, 1] is usually too short. Thus an additional test seems to be needed for a longer sampling range.

In this paper, a known transform pair (Gradshteyn and Ryzhik, 1965)

$$f(r)=\exp(-r^2a)/(2\pi), \quad (6)$$

and

$$F(\rho)=\exp(-\pi^2r^2/a)/(2a). \quad (7)$$

is used to test the AFM described in the last section. Figs. 1 and 2 show the function $f(r)$ of (6) for $a=10$ and its Hankel transformed function $F(\rho)$ of (7), respectively. In the numerical experiment, the function $f(r)$ is sampled uniformly on an interval [0, 5] in both 128 and 512 samples, so accuracy can be compared for the two sampling rates.

Fig. 3 shows errors of the numerical Hankel transforms with 128 and 512 samples. The maximum departures from theory occur at $\rho=0$ in both samplings, and they are 8.2×10^{-5} in 128 samples and -3.3×10^{-5} in 512 samples, respectively. Root mean squares (RMS) errors

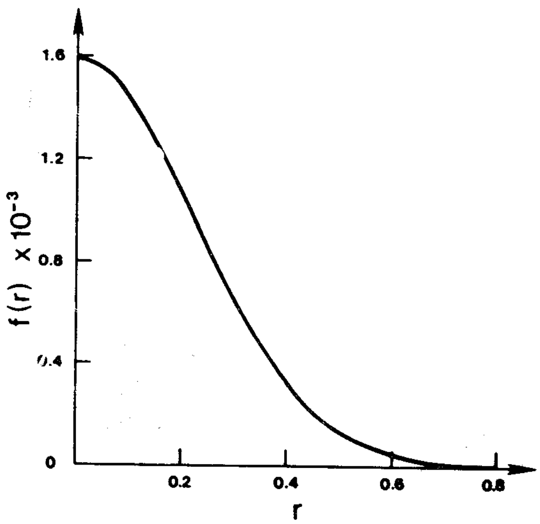

Fig. 1 Test function, $f(r)=\exp(-r^2a)/(2\pi)$ where $a=10$, to be Hankel transformed.

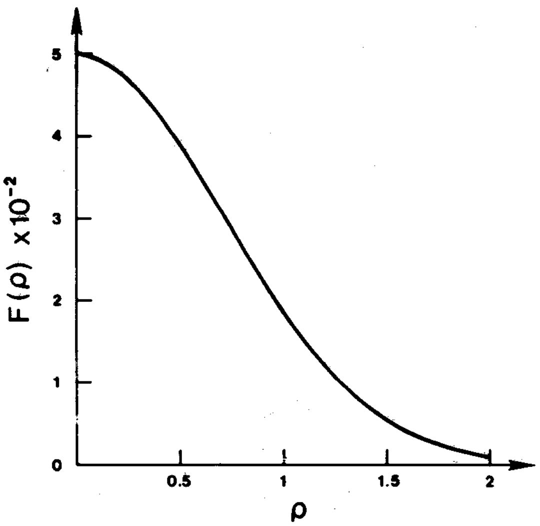

Fig. 2 Hankel transform of the test function shown in Fig. 1.

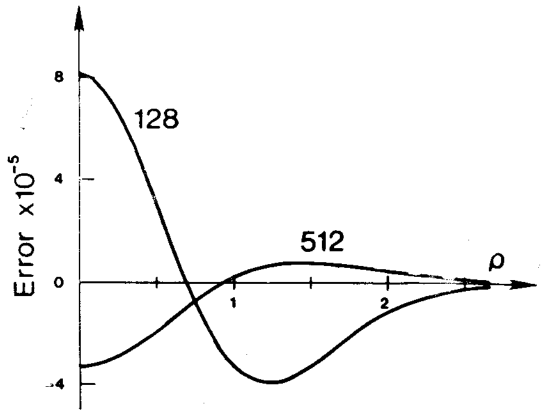

Fig. 3 Error of numerical Hankel transform computed by Abel-Fourier method.

are respectively 1.9×10^{-5} in 128 samples and 7.5×10^{-6} in 512 samples as shown in Table 1. These results show that a denser sampling reduces the numerical error.

Anderson's DLFM produces more accurate Hankel transform than the AFM for the same test function $f(r)$. Table 1 shows that agreement with theory in the DLFM is better than that in the AFM. In the Anderson's filtering, an average number of function evaluations was about 53.

Another examples to test the AFM is picked

Table 1 Errors of numerical Hankel transforms computed by Abel-Fourier method (AFM) and by Anderson's filtering method (DLFM).

	Peak error (ρ)	RMS error
AFM		
128 samples	8.2×10^{-5} (0.0)	1.9×10^{-5}
512 samples	-3.3×10^{-5} (0.0)	7.5×10^{-6}
DLFM	0.6×10^{-6} (2.0)	1.7×10^{-7}

up from a computation of mutual coupling ratio for 3-layered earth model. The mutual coupling ratio of horizontal coplanar loops, Z/Z_0, is given by (Anderson, 1979)

$$Z/Z_0=1+B^3T_0, \qquad (8)$$

where

$$T_0=\int_0^\infty [R(g)ge^{-gA}]J_0(gB)gdg, \qquad (9)$$

where symbols $R(g)$, A and B are explicitly listed in Kim (1987). Parameters of the 3-layered model are

$\rho_1=100\Omega\cdot$m, $\rho_2=5\Omega\cdot$m, $\rho_3=100\Omega\cdot$m,
$h_1=100$m, and $h_2=10$m,

and a transmitting frequency is 1,000Hz.

Fig. 4 shows the kernel function in (9) when $A=\pi/5$, and Fig. 5 shows the real part of T_0 obtained from the Anderson's DLFM. A peak

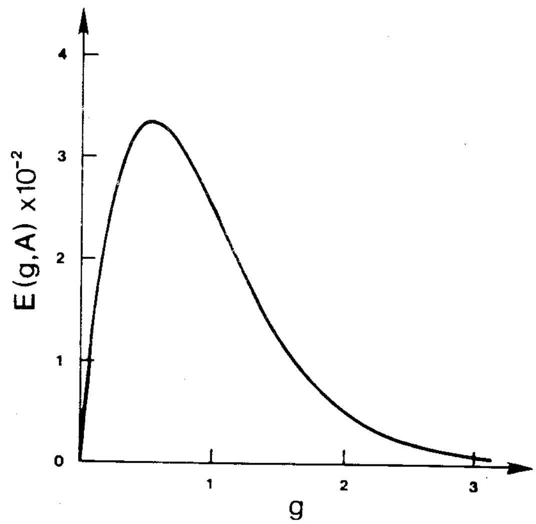

Fig. 4 Electromagnetic kernel function, $E(g,A)=gR(g)\exp(-gA)$ where $A=\pi/5$, to be Hankel transformed.

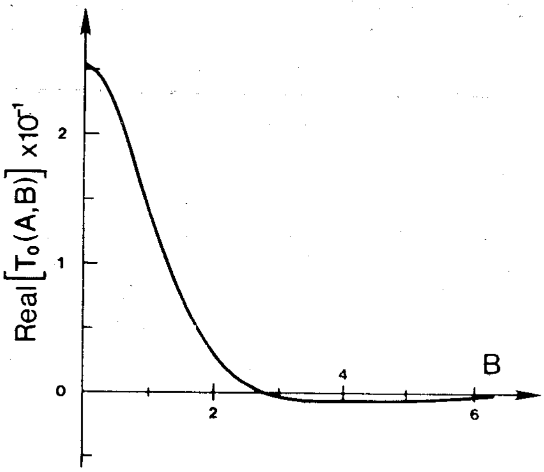

Fig. 5 Real part of T_0 in (9) computed by Anderson's filtering method.

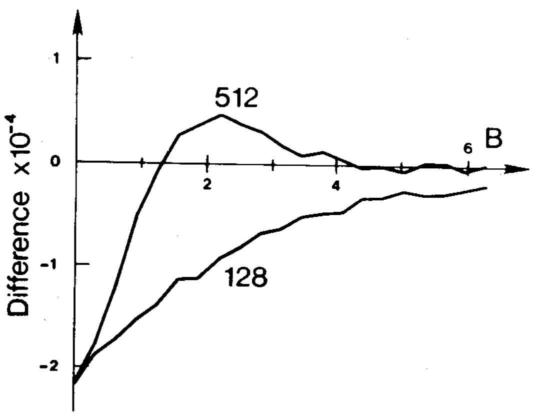

Fig. 6 Differences between results computed by Abel-Fourier method and by Anderson's filtering method.

value of the kernel function appears at about $g=0.5$. In the AFM, the kernel function is sampled uniformly on the interval [0, 10] in both 128 and 512 samples.

Fig. 6 shows differences between results obtained from the AFM and from the DLFM. The difference for 512 samples is smaller than that for 128 samples. RMS differences in 128 and 512 samples are 4.8×10^{-5} and 3.3×10^{-5}, respectively. Negative maxima in the differences occur at $B=0$ in both samplings, and are 2.2×19^{-4}.

DISCUSSION AND CONCLUSIONS

The AFM for computing Hankel transform of a function is based on the fundamental relationship that Hankel transform is Fourier transform of Abel transform. In this paper, the Abel transform is computed by the shift variant recursive filter with order 8, and the Fourier transform is performed by the split-radix FFT (Sorensen et al., 1986).

As expected, an increase of data points in the AFM decreases numerical error (Fig. 3 and Table 1). To improve the results for lower numbers of samples, the scale must be adjusted to place more samples in the "important" part of the function. For example, changing the constant a from 10 to 5 in (6) broaden the function which is equivalent to sampling more densely, and reduces the error significantly.

The AFM is less accurate than the Anderson's DLFM as shown in Table 1. The AFM, however, is computationally more efficient than the DLFM. The AFM requires $O\ (3KN+2N\log N)$ real operations to process N data points, whereas the DLFM requires $O\ (\bar{K}N)$ real operations in which $\bar{K}$ is the average of filter coefficients:

$$\bar{K}=\sum_{n=1}^{N} K_n/N,$$

where K_n is the number of filter coefficients. In the simulation done with known transform pair of (6) and (7), the $\bar{K}$ is about 53. Therefore, the needed operations are slightly smaller in the AFM than in the DLFM.

These operations, however, do not include evaluations of the function to be Hankel transformed for various r's. In principle, both DLFM and AFM are schemes that compute a Hankel transform for a certain value ρ. But the AFM can compute Hankel transforms for various ρ's because of the use of FFT. In examples shown in this paper, total number of compution of Hankel transform depends mainly on the function evaluation in the DLFM, but not so in the AFM. In fact, the needed number of function evaluations in the DLFM is $\bar{K}N$, whereas that in the AFM is only N.

Although the AFM is fast enough to compute Hankel transforms for various ρ's, it is not flexible in a case of computing a Hankel transform for a certain ρ. Even in this case, the AFM cannot reduce total computational efforts. Furthermore, since the AFM uses FFT, it is inconvenient to assign a specific value of ρ in the numerical calculation. Therefore, it may be concluded that the DLFM is more flexible than the AFM in the practical point of view.

In geophysics, there are many problems which use Hankel transforms of order 1. Higher order Hankel transforms may, in principle, be computed by using the projection-slice theorem. But there are no rapid implementation for computing the Hankel transform of order 1 yet.

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Abel-Fourier법과 디지탈 선형필터법과의 비교

김 희 준

요약 : 0차의 Hankel 변환은 Abel 변환과 Fourier 변환의 결합으로 나타낼 수 있다. 이 Abel-Fourier법은 샘플링 간격이 일정하지 않는 중합필터를 사용한 신속한 Abel 변환과 고속 Fourier 변환으로 작성될 때, 종래의 디지탈 선형필터법보다 계산시간면에서 유리하다. 그러나, Abel-Fourier법은 일반적으로 잘 설계된 디지탈 필터보다 정확하지는 않다. 전기탐사 문제에 이들 방법을 적용할 때, 디지탈 필터법이 Abel-Fourier 법보다 더 융통성이 많은 것으로 생각된다.