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Weak Association of Random Variables, with Applications*

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Abstract

Random variables $X_1, X_2, \ldots, X_m$ are said to be weakly associated if whenever π is a permutation of $\{1,2,\ldots,m\}$, $1\leq k<m$, and f: $\boldsymbol{R}^k \longrightarrow \boldsymbol{R}$, g: $\boldsymbol{R}^{m-k} \longrightarrow \boldsymbol{R}$ are coordinatewise nondecreasing functions then $Cov\,[f(X_{\pi(1)},\ldots, X_{\pi(k)}),\ g(X_{\pi(k+1)},\ldots,X_{\pi(m)})]\geq 0$, whenever the covariance is defined. An infinite collection of random variables is weakly associated if every finite subcollection is weakly associated. The basic properties of weak association and central limit theorem for weakly associated random variables are derived. We also extend this idea to point random fields and prove that a *Cox* process with a stationary weakly associated intensity rardom measure is weakly associated. Another inequalities and the fact that positive correlated normal random variables are weakly associated are also proved.

1. Introduction

The concept of associated random variables was introduced into the statistical literature by Esary, Proschan, and Walkup [3]. Since then a great many papers have been written on the subject and its extensions and numerous multivariate inequalities have been obtained [2], [3], [4], [8], [9].

In addition to this Joag-Dev and Proschan introduced the notion of negatively associated random variables, derived basic theoretical properties, and developed applications in multivariate statistical analysis [5]. In this paper we introduce the notion of weakly associated random variables, derive theoretical results, and obtain some applications.

In section 2 we define weakly associated, develop its basic properties, and prove a central limit theorem for weakly associated random variables with Newman's idea

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[7], [8]. Some applications are given in section 3, moment inequalities, positively correlated normal random variables are weakly associated, and a *Cox* process with weakly associated intensity random measure is weakly associated.

In sequal, we introduce some definitions:

Definition 1.1. A collection $\{X_i: i\in I\}$ of random variables is called associated if for every finite subcollection $X_1,\ldots,X_m$ and every pair of coordinatewise nondecreasing functions f_1, f_2: $\mathbf{R}^m\longrightarrow\mathbf{R}$, we have that the random variables $\tilde{f}_j=f_j(X_1,\ldots,X_m)$, $j=1,2$, satisfy

$Cov\ (\tilde{f}_1,\tilde{f}_2)\geq 0$ whenever they are such that $E(\tilde{f}^2{}_j)<\infty$ for $j=1,2$ ([2], [3]).

Definition 1.2. Random variables $X_1, X_2,\ldots,X_m$ are said to be negatively associated if for every pair of disjoint subsets A_1, A_2 of $\{1,2,\ldots,k\}$,

$$Cov\ (f_1(X_i;\ i\in A_1),\ f_2(X_j;\ j\in A_2))\leq 0$$

whenever f_1: $\mathbf{R}^{A_1}\longrightarrow\mathbf{R}$, f_2: $\mathbf{R}^{A_2}\longrightarrow\mathbf{R}$ are nondecreasing. An infinite family of random variables is negatively associated if every subfamily is negatively associated. Also, without loss of generality, we may assume that $A_1\cup A_2=\{1,\ldots,k\}$, [5].

2. Basic Definitions and Results.

Definition 2.1. Random variables $X_1, X_2,\ldots,X_m$ are said to be weakly associated if whenever π is a permutation of $\{1,2,\ldots,m\}$, $1\leq k<m$, and f: $\mathbf{R}^k\longrightarrow\mathbf{R}$, g: $\mathbf{R}^{m-k}\longrightarrow\mathbf{R}$ are coordinatewise nondecreasing functions then

$$Cov(f(X_{\pi(1)},\ldots,X_{\pi(k)}),\ g(X_{\pi(k+1)},\ldots,X_{\pi(m)})\geq 0 \cdots\cdots\cdots (2.1)$$

whenever the covariance is defined. An infinite collection of random variables is weakly associated if every finite subcollection is weakly associated (private communication with Burton).

The above definition can be written as follows:

Random variables $X_1,\ldots,X_k$ are said to be weakly associated if for every pair of disjoint subsets A_1, A_2 such that $A_1\cup A_2=\{1,2,\ldots,k\}$,

$$Cov(f(X_i,\ i\in A_1),\ g(X_j,\ j\in A_2))\geq 0 \cdots\cdots\cdots (2.2)$$

whenever f and g are nondecreasing.

Proposition 2.2. Suppose $\tilde{N}=(N_1, N_2, ..)$ is associated with values in Z^+ and X_1, $X_2, \ldots$ is independent conditioned on N with distribution $F_{N1}, F_{N2}, \ldots$. Then $X_1, X_2, \ldots$ is weakly associated.

Proof. Let π be a permutation of $\{1, \ldots, m\}$, $1 \leq k < m$, and $\tilde{f}=f(X_{\pi(1)}, \ldots, X_{\pi(k)})$, $\tilde{g}=g(X_{\pi(k+1)}, \ldots, X_{\pi(m)})$, where f, g are coordinatewise nondecreasing. Then

$$Cov(\tilde{f}, \tilde{g})=E[Cov(\tilde{f}|N, \tilde{g}|N)+Cov(E(\tilde{f}|N), E(\tilde{g}|N)) \cdots\cdots\cdots (2.3)$$

Now for fixed value of N, $Cov(\tilde{f}|N, \tilde{g}|N) \geq 0$ since $X_1, X_2, \ldots$ are N-conditionally weakly associated so the first term on the right hand side is nonnegative. Also $E(\tilde{f}|N)$ and $E(\tilde{g}|N)$ are nondecreasing functions of N, [3]. So the second term on the right hand side is also nonnegative because N is associated.

The following three properties are obvious from the above definition.

Property P1. A subset of weakly associated random variable is weakly associated.

Property P2. A set of independent (or associated) random variables is weakly associated.

Property P3. Nondecreasing functions defined on disjoint subsets of a set of weakly associated random variables are weakly associated.

Property P4. The union of independent sets of weakly associated random variables is weakly associated.

Proof. Let X, Y be independent vectors, each weakly associated. We shall show that the vector (X, Y) is weakly associated. Let (X_1, X_2) and (Y_1, Y_2) denote arbitrary partitions of X and Y respectively. Let f and g be arbitrary nondecreasing functions. Note that $E[f(X_1, Y_1)|Y_1]$ is a Y_1 measurable function, so that

$$E[f(X_1, Y_1)|Y_1, Y_2]=E[f(X_1, Y_1)|Y_1] \cdots\cdots\cdots (2.4)$$

A similar result holds for $E[g(X_2, Y_2)|Y_2]$. Denote these conditional expectaions by $h_1(Y_1)$ and $h_2(Y_2)$ respectively and note that h_1, h_2 are nondecreasing. Thus

$$\begin{aligned} E[f(X_1, Y_1)\ g(X_2, Y_2)] &= E[E\{f(X_1, Y_1)\ g(X_2, Y_2)|Y_1, Y_2\}] \\ &\geq E[h_1(Y_1) \cdot h_2(Y_2)] \\ &\geq E[h_1(Y_1)]\ E[h_2(Y_2)] \\ &= E[f(X_1, Y_1)]\ E[g(X_2, Y_2)], \end{aligned}$$

where the first inequality follows from the fact that (X_1, X_2) is independent of (Y_1, Y_2) and hence weakly associated is preserved under conditioning. The second inequality holds since (Y_1, Y_2) is weakly associated.

Remark. Properties P3 and P4 broaden the scope of application of weak association considerably.

Definition 2.3. (Lehmann, 1966) Random variables X and Y are positive quadrant dependent if for every real x, y,

$$P(X \leq x, Y \leq y) \geq P(X \leq x)\ P(Y \leq y). \quad \cdots\cdots (2.5)$$

Property P5. For a pair of random variables, X, Y are positively quadrantdependent if and only if X, Y are weakly associated.

This follows immediately from Definitions 2.1 and 2.3.

Property P6. Let $A_1, \ldots, A_m$ be disjoint subsets of $\{1, 2, \ldots, k\}$ and $f_1, f_2, \ldots, f_m$ be nondecreasing positive functions. Then for $X_1, \ldots, X_k$ weakly associated

$$E \Pi^m_{i=1} f_i(X_j,\ j \in A_i) \geq \Pi^m_{i=1} E\ f_i(X_j,\ j \in A_i). \quad \cdots\cdots (2.6)$$

This follows from the repeated application of Definition 2.1.

Property 7. An immediate consequence of Property P6 is that for A_1, A_2 disjoint subsets of $\{1, 2, \ldots, k\}$, and $x_1, \ldots, x_k$; real,

$$P(X_i \leq x_i,\ i=1, \ldots, k) \geq P(X_i \leq x_i,\ i \in A_1)\ P(X_j \leq x_j,\ j \in A_2) \quad \cdots (2.7)$$

and

$$P(X_i > x_i,\ i=1, 2, \ldots k) \geq P(X_i > x_i,\ i \in A_1)\ P(X_j > x_j,\ j \in A_2). \cdots (2.8)$$

Let $X_1, \ldots, X_m$ be a sequence of real random variables and $S_n = \sum_{i=1}^{n} X_i$.

Lemma 2.4. Suppose that $X_1, X_2, \ldots$ are weakly associated then for every $k > n$ and non decreasing functions f and g,

$$Cov[f(S_n),\ g(S_k - S_n)] \geq 0. \quad \cdots\cdots (2.9)$$

Proof. This follows immediately from Definition 2.1 and Property P3. ///

The proof of Theorem 12 of [8] proves the following theorems [7].

Theorem 2.5. Let $\{X_i,\ i \geq 1\}$ be a strictly stationary, finite variance sequence such that

$$\lim_{n\to\infty} \text{Var}\left(\frac{S_n}{\sqrt{n}}=\right)\sigma^2<\infty. \quad\cdots\cdots (2.10)$$

and satisfy (2.9).

Then

$$\frac{1}{\sqrt{n}}(S_n - n\ EX_1)\xrightarrow{a.s.} n(0,\sigma^2).$$

Lemma 2.4, and Theorem 2.5. yield following a central limit theorem for weakly associated random variables;

Theorem 2.6. Let $X_1, X_2, \ldots$ be a strictly stationary finite variance, sequence of weakly associated random variables satisfying (2.10).

Then $X_1, X_2, \ldots$ satisfy a central limit theorem.

3. Applications.

3.1. Cox process Let $\wedge$ be a stationary, weakly associated random measure and let Z be a doubly stochastic Poisson point random point (Cox process) with environment $\wedge$; that is Z is conditional Poisson process with intensity measure $\wedge$. Then Z is weakly associated [2].

Proof. Let $\tilde{f}=f(Z(B_1),\ldots,Z(B_k))$, $\tilde{g}=(Z(B_{k+1}),\ldots,Z(B_n))$, where f, g are coordinatewise nondecreasing. We need to show

$$Cov(\tilde{f},\tilde{g})=E(Cov(\tilde{f}\mid\wedge,\ \tilde{g}\mid\wedge))+Cov(E(\tilde{f}\mid\wedge),\ E(\tilde{g}\mid\wedge))\geq 0.$$

Now for fixed value of $\wedge$ $Cov(\tilde{f}\mid\wedge, \tilde{g}\mid\wedge)\geq 0$ since $Z(B_1)$, $Z(B_2),\ldots$ are $\wedge$-conditionally associated (*i.e.* $Z(B_1)$, $Z(B_2),\ldots$ are $\wedge$-conditional Poisson with intensity measure $\wedge$) so the first term of the sum is nonnegative. Also $E(\tilde{f}\mid\wedge)$ and $E(\tilde{g}\mid\wedge)$ are nondecreasing functions on $\wedge$ so the second term is also nonnegative because $\wedge$ is weakly associated. ///

3.2 Moment in equalities. Property P6 can be utilized to derive moment inequalities. Suppose $X_1,\ldots, X_m$ are weakly associated positive random variables $\alpha_i\geq 0$, $i=1,\ldots,k$ with $k\leq m$. Then

$$E(X_1^{\alpha_1}, X_2^{\alpha_2},\ldots,X_k^{\alpha_k}\geq \Pi_{i=1}^k\ E(X_i^{\alpha_i})$$

In particular, $E(X_1, X_2, \ldots, X_k) \geq \Pi^k E(X_i)$

In the above inequalities $X_1, X_2, \ldots, X_k$ could be replaced by any other subset of k choosen from $X_1, \ldots, X_m$.

3.3. Positively correlated normal random variables are weakly associated.

We use the same approach as in the proof that negatively correlated normal variables are negative associated given by Joag-Dev and Proschan. [5]. The present case is actually simpler than the proof that positively correlated normal random variables are associated given by Joag-Dev, Perlman, and pitt (1982). Since the functions utilized are defined on disjoint sets.

Proof. Let $X_1, \ldots, X_k$ be jointly normally distributed random variables with covariance matrix $\Sigma = (\sigma_{ij} = Cov[X_i, X_j])$. By a lemma of Slepian (1962), it follows that

$$\frac{\partial}{\partial \sigma_{12}} P\{(X_1 \geq a_1(x_3),\ X_2 \geq a_2(x_3),\ x_3 \in A\} \geq 0. \cdots\cdots\cdots\cdots\cdots (3.1)$$

where $x_3 = (X_3, X_4, \ldots, X_k)$, a_1, a_2, are defined on R^{k-2}.

A is an arbitrary measurable set in R^{k-2}, and σ_{ij} is the element in the i-th row and j-th column of Σ. From (3.1) it follows that $f_1(X_1, x_3)$ and $g_1(X_2, x_3)$ are non-decreasing in X_1 and X_2 respectively, then

$$E[f(X_1, x_3),\ g(X_2, x_3)] \uparrow \text{in } \sigma_{12}. \cdots\cdots\cdots\cdots\cdots (3.2)$$

Suppose now that f and g are nondecreasing in each argument and A_1, A_2 are disjoint subsets of $\{1, \ldots, k\}$. Then

$$E[f(x_j,\ j \in A_1)\ g(x_1,\ i \in A_2)] \uparrow \text{ in } \sigma_{ij} \cdots\cdots\cdots\cdots\cdots (3.3)$$

for every pair (i, j) such that $i \in A_1$ and $j \in A_2$. From (3.3), it follows that if $\sigma_{ij} \geq 0$ for every pair (i, j), then

$$E[f(x_i,\ i \in A_1)\ g(x_j,\ j \in A_2)] \geq E(f(x_i,\ i \in A_1))\ E(g(x_j, j \in A_2)).$$

The desired result is now proven. ///

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