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Numbers of Generators of Maximal Ideals in Polynomial Rings

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1. Introduction

The cardinality of a minimal basis of an ideal I is denoted $\nu(I)$. Let A be a polynomial ring in $n>o$ variables with coefficients in a Noetherian(commutative with $1\neq 0$) ring R, and let M be a maximal ideal of A. In general, $\nu(MA_M)\leq\nu(M)\leq\nu(MA_M)+1$. In many cases, the lower bound is attained.

In [3], it is shown that equality is attained in each of the following cases: (1) A_M is not regular, (2) $M\cap$R is maximal in R and (3) $n>1$. Hence the problem of determining whether $\nu(M)=\nu(MA_M)$ can be studied when $M\cap R$ is not maximal, A_M Is regular and $n=1$. The purpose of this paper is to provide some conditions in which the lower bound is satisfied, when $n=1$, A_M is regular and $M\cap R$ is not maximal.

2. Preliminaries

Definition 1. An integral domain R satisfying the following eguivalent conditions is called a Dedekind domain

1) Every non-zero ideal of R is invertibe
2) R is Noetherian, integrally closed, and dimension≤ 1

Definition 2. A super-regular ring is a Noetherian ring such that R_P is a regular for every prime ideal in R.

Definition 3. Local ring S is complete intersection if $S=R/I$, where R is regular local ring and I is generated by a regular sequence.

Lemma 1. Let $A=R[X_1, X_2, \cdots, X_n]$ with a maximal ideal M and $M\cap R=P$. Then R/P is a G-domain.

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Proof. $A/PA = R[X_1, \cdots, X_n]/PR[X_1, \cdots, X_n] \simeq R/P[X_1, \cdots, X_n]$. Since M/PA is a maximal adeal of A/PA such that $M/PA \cap R/P = 0$. By [4, Theorem 24] R/P is G-domain. ///

Lemma 2. A prime ideal I is generated by a regular sequence if and only if $\nu(I) = ht(I)$.

Proof. [2]

Lemma 3. Let R be a Dedakind domain with quotient field K. Let L be a finite-dimensional over K, and let T be the integral closure of R in L. Then T is a Dedekind domain.

Proof. [4, Theorem 98]

Lemma 4. D is a UFD and $K(\theta)$ is a quadratic extension field of K, where $\theta^2 \varepsilon D$, θ^2 not divisible by the square of a prime in D. Ler J be the integral closure of D in $K(\theta)$.

If char $D \neq 2$, $J = \{a + b\theta \mid a,\ b \varepsilon K,\ 2a \varepsilon D,\ 2b \varepsilon D,\ a^2 - b^2\theta^2 \varepsilon D\}$

if char $D = 2$, $J = \{a + b\theta \mid a,\ b \varepsilon K,\ a^2 - b^2\theta^2 \varepsilon D\}$

if $D = Z$, $J = \{a + b\theta \mid a,\ b \varepsilon Z\}$ if $\theta^2 \not\equiv 1 \ (mod\ 4)$

while, $J - \{(a + b\theta)/2 \mid a,\ b \varepsilon Z,\ a \equiv b \ (mod\ 2)\}$ if $\theta^2 \equiv 1 \ (mod\ 4)$

Proof. [7, p. 100]

Lemma 5. Let I be an ideal of a ring R. Suppose that pro. dim $I < \infty$ and I/I^2 is free over R/I. If $\nu(I) = \nu(I/I^2)$ then I is generated by a regular sequence.

Proof. [1, proposition 1]

Lemma 6. Let R be a Noetherian ring. Let $J \subset I$ be two ideals of R with $\mathscr{A}(I) = \mathscr{A}(J)$, and let $\nu(I/J) = m$. Further let $p_1, \cdots, p_s \varepsilon Spec(R)$ with $I \not\subset \bigcap_{j=1}^{s} P_j$ be given. Then one can find elements $a_1, \cdots, a_m \varepsilon I$ such that:

a) $I = (a_1, \cdots, a_m) + J$

b) $a_i \notin \bigcap_{j=1}^{s} p_j \ (i = 1, \cdots, m)$

c) *If* $p \varepsilon \mathscr{A}(a_1, \cdots, a_m)$, $P \notin \mathscr{A}(I)$, then $ht(p) \geq m$.

In this Lemma notation $\mathscr{B}(I)$ means $\mathscr{B}(I)=\{P\varepsilon Spec(R)\,|\,P\supset I\}$

Proof. [5, p. 142]

3. Theorems

We consider the case, the lower bound is not satisfied, when $n=1$, A_M is regular and $M\cap R$ is not maximal. Z be a integer ring is a Dedekind domaih with quotient field Q Since $X^2+5=0$, $\sqrt{5}i$ is a algebraic over Q. Hence $Q(\sqrt{5}i)$ be a finite dimensional field extension over Q. By Lemma 4, $Z[\sqrt{-5}]$ is the integral closure of Z in $Q(\sqrt{5}i)$. Hence by Lemma 3, $Z[\sqrt{-5}]$ is a Dedekind domain and not a UFD. By definition 1, $Z[\sqrt{-5}]$ is a Noetherian and dim $Z[\sqrt{-5}]\leqq 1$. We consider $p=\langle 3,\ 1+2\sqrt{-5}\rangle$ in $Z[\sqrt{-5}]$. Consider $P\cap Z$. We have $3\varepsilon P\cap Z$, and if any integer not divisible by 3 lies in $P\cap Z$ then the form is $3n+1$, where $n\varepsilon Z$, $1\varepsilon P\cap Z$ and we have the contradiction $p=Z[\sqrt{-5}]$. Hence we see that $P\cap Z=3Z$. Given $x, y\varepsilon Z[\sqrt{-5}]$ let u, v be integers such that $x\text{-}u\varepsilon P$, $y\text{-}v\varepsilon P$. Suppose $xy\varepsilon P$. Then $uv\in P$ and, since $uv\varepsilon Z$, we have $uv\varepsilon P\cap Z=3Z$. Thus $u\varepsilon 3Z$ or $v\varepsilon 3Z$, so either $x\varepsilon P$ or $y\varepsilon P$. Hence P is prime. Therefore dim $Z[\sqrt{-5}]=1$ and dim $Z[\sqrt{-5}][X]=2$. Let $R=Z[\sqrt{-5}]$. $A=R[X]=Z[\sqrt{-5}][X]$. $M=<\{\sqrt{5}iX-1,\ aX-1\,|\,a\varepsilon Z\}>$. Since A/M is a fied, M is a maximal in A and $M\cap R=0$ not maximal in R. Since any Dedekind domain is super-regular, A_M is a regular. And since $P[X]\subseteq M$, $\nu(MA_M)=ht(M)=1$. $\nu(M)$ is at least 2 and $1=\nu(MA_M)\leq\nu(M)\leq\nu(MA_M)+1=2$. Hence $\nu(M)=2>\nu(MA_M)=1$.

From now on, let $A=R[X]$, where R is a Noetherion. commutative ring with $1\neq 0$ and M is a maximal ideal of A such that $M\cap R=p$ is not maximal ideal of R and dim $R=d$.

Theorem 1. Let R de a regular local ring with the maximal ideal $\underline{m}$. Let R/p is a complete intersection. If $\underline{m}^2\cap p=p\underline{m}$, then $\nu(M)=\nu(MA_M)$.

Proof. Since $\underline{m}^2\cap p=\underline{m}p$, $\nu(\underline{m})=\nu(p)+\nu(\underline{m}/p)$, by [8, p. 27]. And since R/p is a complete intersection, p is generated by a regular sequence, $\nu(p)=ht(p)$. Because R is regular, R is Cohen-Macaulay ring, $ht(\underline{m}/p)=ht(\underline{m})-ht(p)$. Since R is regular and $\nu(p)=ht(p)$, $\nu(\underline{m}/p)=\nu(\underline{m})-\nu(p)=ht(\underline{m})-ht(p)$. We obtain $\nu(\underline{m}/p)=ht(\underline{m}/p)$. Hence R/p Is a regular ring. Therefore $(R/p,\ \underline{m}/p)$ is a regular local ring, R/p is

UFD by [4, Theorem 184]. But $M/pR[X]\subset A/pA=R/p[X]$ is UFD, since $ht(M)=ht(p)+1=(d-1)+1=d$. The last two equality is that R/p is G-domain dim $R/P\leq 1$. By hypothesis p is not maximal, dim $R/p=1$ and $ht(p)=d-1$. And $ht(pR[X])=ht(p)=d-1$, hence $M/pR[X]$ is an ideal of ht 1 in UFD. Hence $\nu(M/pR[X])=1$ by [6, Theorem 47]. therefore $\nu(M)\leq\nu(pR[X])+\nu(M/pR[X])\leq\nu(p)+\nu(M/pR[X])=ht(p)+1=\nu(pRp)+1=\nu(MA_M)$. Last equality is by [3, Lemma 3]. Hence $\nu(M)=\nu(MA_M)$. ///

Proposition. For an ideal I of a Noetherian ring R,

$$ht(I)\leq\nu(I/I^2)\leq\nu(I)\leq\nu(I/I^2)+1.$$

Proof. For all $P\supset I$. Let $I_p/I_p{}^2=<\bar{m}_1,\cdots,\bar{m}_t>$, where t = minimal number of generators of $I_p/I_p{}^2$. Then $I_p=<m_1,\cdots,m_t>+I_p{}^2=<m_1,\cdots,m_t>+I_p\cdot I_p$. By Nakayama's Lemma $I_p=<m_1,\cdots,m_t>$. Hence $\nu(I_p)\leq\nu(I_p/I_p{}^2)$. Therefore $\nu(I_p)=\nu(I_p/I_p{}^2)$ for all $P\supset I$. In general, $ht(I)=ht(I_p)\leq\nu(I_p)=\nu(I_p/I_p{}^2)\leq\nu(I/I^2)\leq\nu(I)$. To show that $\nu(I)\leq\nu(I/I^2)+1$. Consisder elements $m_1,\cdots,m_r\varepsilon I$. whose residue classes in I/I^2 from a minimal generating system. Let $R^*=R/<m_1,\cdots,m_r>$ and $I^*=I/<m_1,\cdots,m_r>\cdot(I^*)^2=(I/<m_1,\cdots,m_r>)^2=I^2+<m_1,\cdots,m_r>/<m_1,\cdots,m_r>=I/<m_1,\cdots,m_r>=I^*$. The last two equality is $I/I^2=<\bar{m}_1,\cdots,\bar{m}_r>$. Then I^* is principal. For $I^*=<a_1,\cdots,a_s>$, s = minimal number of generators of I^*. Since $I^*=(I^*)^2$, $a_i=\sum_{k=1}^{s}r_{ik}a_k$, where $r_{ik}\varepsilon I^*$. Hence $\sum_{k=1}^{s}(\delta_{ik}-r_{ik})a_k=0$. Multiply by adjoint$(\delta_{ik}-r_{ik})$. Then $\det(\delta_{ik}-r_{ik})$. $a_k=0(k=1,\cdots,s)$. $\det(\delta_{ik}-r_{ik})=1-a$, $\exists\, a\varepsilon I^*$. Since a is a linear combination of the a_k. It follows from $\det(\delta_{ik}-r_{ik})\cdot a_k=0$ that $(1-a)\cdot a=0$ so $a^2=a$. From $(1-a)\cdot a_k=0$. We get $a_k=a_k\cdot a\varepsilon<a>$, so $I^*=(a)$. Hence I is generated by $r+1$ elements. ///

Theorem 2. Let A_M is a regular ring. If $\nu(M/M^2)>\dim R+1$ then $\nu(M)=\nu(MA_M)$.

Proof. Let $t=\nu(M/M^2)$. We first show that $\nu(M)\leq t$. By Lemma 6, we can choose elements $a_1,\cdots,a_t$ so that every $\mathscr{P}\varepsilon\mathscr{B}(a_1,\cdots,a_t)$ with $\mathscr{P}\notin\mathscr{B}(M)$ has height $\geq t$. Since dim $A<t$, $ht\ \mathscr{P}<t$. Hence $\mathscr{B}(a_1,\cdots,a_t)-\mathscr{B}(M)=\phi$ that is, $(M/<a_1,\cdots,a_t>)^n=M^n+<a_1,\cdots,a_t>/<a_1,\cdots,a_t>=0$, there exist positive integer n. Since $(M/<a_1,\cdots,a_t>^2=M/<a_1,\cdots,a_t>$. By the same method of $M/<a_1,\cdots,a_t>=<e>$, where $e^2=e$, $e\varepsilon M$. Hence $(M/<a_1,\cdots,a_t>)^n=M/<a_1,\cdots,a_t>$ for all $n\geq 1$. Therefore $M/<a_1,\cdots,a_t>=0$. Then $\nu(M)\leq t$ and furthermore $\nu(M)=\nu(M/M^2)$. Since *pro. dim* M = *pro. dim* $MA_M\leq gl.$

dim $A_M < \infty$ for A_M is regular. And M/M^2 is free over A/M. By Lemma 5, M is generated by a regular sequence. We have $ht\ M = ht\ MA_M = \nu(MA_M) \leq \nu(M) = ht\ M$. Hence $\nu(M) = \nu(MA_M)$.

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