HONAM MATHEMATICAL JOURNAL
Volume 9, Number 1. August 1987

Fixed Point Theorems for Expansion Mappings on Probabilistic Metric Spaces

by

B.D. Pant*, R.C. Dimri* and S.L. Singh**

* *Department of Mathematics Govt. Postgraduate College Gopeshwar, Chamoli 246 401 India.*

** *Department of Mathematics Gurukul Kangri University Gurukul Kangri, Haridwar 249 404 India.*

Abstract

In this paper we introduce the notion of expansion mapping on a probabilistic metric space and prove common fixed point theorems for a pair of such mappings. These results being new of their kind, are interesting generalizations of some of the known results.

1. Introduction

Some fixed point theorems for expansion mappings on a metric space have been proved in [5]. Recently, Rhoades [1] has generalized these results for a pair of mappings. While a number of fixed point theorems for [1]~[4] contraction mappigs on probabilistic metric spaces have been established in recent years (for an extensive bibliography, see [4]) but fixed point theorems for expansion mappings on probabilistic metric spaces established here are the first to appear in the literature.

2. Preliminaries

First we give some definitions and notations to be usad in the sequel.

Definition 2.1. A probabilistic metric space(PM-space) is an ordered pair$(S, \mathscr{F},)$ where S is a nonempty set and $\mathscr{F}$ is a mapping of $S\times S$ into $\mathscr{L}$, the set of distribution functions. The value of $\mathscr{F}$ at $(u,v)\in S\times S$ will be denoted by $F_{u,v}$ and $F_{u,v}$ are assumed

Received November 25. 1986. Accepted June 24, 1987.

to satisfy the following conditions:

(a) $F_{u,v}(x)=1$ for all $x>0$ iff $u=v$;

(b) $F_{u,v}(0)=0$;

(c) $F_{u,v}=F_{v,u}$;

(d) if $F_{u,v}(x)=1$, $F_{v,w}(y)=1$ then $F_{u,w}(x+y)=1$.

Definition 2.2. A Menger space is a triplet $(S, \mathscr{F}, t)$ where $(S, \mathscr{F})$ is a PM-space and t is the t-norm [2] such that

$$F_{u,w}(x+y)\geq t\{F_{u,v}(x),\ F_{v,w}(y)\}$$

holds for all u, v, w in S and $x,\ y\geq 0$.

We now introduce the following definitons.

Definition 2.3. A mapping $f:S\to S$ will be called an expansion mapping iff for a constant $h>1$

(e) $$F_{fu,fv}(hx)\leq F_{u,v}(x)$$

for all u, v in S and $x\geq 0$.

The interpretation of (e) is as follows: The probability that the distance between the image points fu, fv is less than hx is never greater than the probability that the distance between u, v is less than x.

Definition 2.4. Two mappings $f, g:S\to S$ will be called an expansion pair (f, g) iff for a constant $h>1$

$$F_{fu,gv}(hx)\leq F_{u,v}(x)$$

for all u,v in S and $x\geq 0$.

3. Results

Theorem 3.1. Let $(S,\mathscr{F},t)$ be a complete Menger space where t is continuous and satisfies $t(x,x)\geq x$ for every $x\in[0, 1]$. If (f, g) is an expansion pair of S onto itself then f and g have a unique common fixed point.

Proof. As in [1] we construct a sequence $\{u_n\}$ with

$$u_{2n+1}\in f^{-1}u_{2n},\ u_{2n+2}\in g^{-1}u_{2n+1},\ n=0,1,2,\cdots$$

Suppose $u_{2n}=u_{2n+1}$ for some n. Then u_{2n} is a fixed point of f and also

(1) $$F_{u_{2n+1},\ u_{2n}}(hx)=1.$$

Since (f, g) is an expansion pair, we have

$$F_{u_{2n+1},\ u_{2n}}(hx)=F_{gu_{2n+2},\ fu_{2n+1}}(hx)\leq F_{u_{2n+2},\ u_{2n+1}}(x),$$

which, in the light of (1), implies $u_{2n+1}=u_{2n+2}$ i.e. u_{2n} is a fixed point of g also. Similarly $u_{2n+1}=u_{2n+2}$ gives that u_{2n+1} is a common fixed point of f and g.

Now suppose $u_{2n}\neq u_{2n+1}$, for each n. Then

$$F_{u_{2n},\ u_{2n+1}}(hx)=F_{fu_{2n+1},\ gu_{2n+2}}(hx)\leq F_{u_{2n+1},\ u_{2n+3}}(x)$$

and

$$F_{u_{2n+1},\ u_{2n+2}}(hx)=F_{gu_{2n+2},\ fu_{2n+3}}(hx)\leq F_{u_{2n+2},\ u_{2n+3}}(x).$$

Thus, in general

$$F_{u_n,\ u_{n+1}}(hx)\leq F_{u_{n+1},\ u_{n+2}}(x),\quad u_n\neq u_{n+1}.$$

So, in view of the lemma [3], $\{u_n\}$ is a Cauchy sequence and converges to a point w in S. Therefore for an integer K such that $n\geq K(\varepsilon,\lambda)$, $\varepsilon>0$, $\lambda>0$,

(2) $$F_{u_{2n+1},\ w}(\varepsilon)>1-\lambda.$$

Let $s\in f^{-1}w$. Then for $u_n\neq w$

$$F_{u_{2n+1},\ w}(\varepsilon)=F_{gu_{2n+2},\ fs}(\varepsilon)\leq F_{u_{2n+2},\ s}(\varepsilon/h),$$

which with the help of (2) gives $s=w$ and so $fs=fw=w$.

Also, if $v\in g^{-1}w$ then,

$$F_{u_{2n},\ w}(\varepsilon)=F_{fu_{2n+1},\ gv}(\varepsilon)=F_{u_{2n+1},\ v}(\varepsilon/h)$$

implies that $w(=v)$ is a common fixed point of f and g.

Uniqueness of the common fixed point can be seen by noting that (f, g) is an expansion pair. ///

Theorem 3.2. Let $(S, \mathscr{F}, t)$ be a complete Menger space where t is continuous and satisfies $t(x,x)\geq x$ for every $x\in[0, 1]$. If f, g are continuous mappings of S onto itself satisfying

(3) $$F_{fu,\,gv}(hx)\leq max\{F_{u,\,fu}(x),\ F_{v,\,gv}(x),\ F_{u,\,v}(x)\}$$

for all u,v in S, $x\geq 0$ and $h>1$, then f or g has a fixed point or f and g have a common fixed point.

Proof. We define $\{u_n\}$ as in the preceding theorem. If $u_n=u_{n+1}$ for any n, then f or g has a fixed point. Suppose $u_n\neq u_{n+1}$ for each n. Then by (3),

$$F_{u_{2n},\,u_{2n+1}}(hx)\leq max\{F_{u_{2n+1},\,u_{2n}}(x),\ F_{u_{2n+2},\,u_{2n+1}}(x),\ F_{u_{2n+1},\,u_{2n+2}}(x)\}$$

and

$$F_{u_{2n+1},\,u_{2n+2}}(hx)\leq max\{F_{u_{2n+3},\,u_{2n+2}}(x),\ F_{u_{2n+2},\,u_{2n+1}}(x),\ F_{u_{2n+3},\,u_{2n+2}}(x)\}.$$

Thus for each n,

$$F_{u_n,\,u_{n+1}}(hx)\leq max\{F_{u_n,\,u_{n+1}}(x),\ F_{u_{n+1},\,u_{n+2}}(x)\}.$$

But, as $h>1$,

$$F_{u_n,\,u_{n+1}}(hx)<F_{u_n,\,u_{n+1}}(x).$$

Therefore

$$F_{u_n,\,u_{n+1}}(hx)\leq F_{u_{n+1},\,u_{n+2}}(x).$$

This implies that $\{u_n\}$ is a Cauchy sequence and converges to some w in S.

Now as $u_{2n}=fu_{2n+1}$, $u_{2n+1}=gu_{2n+2}$ and f, g are continuous, it is standard to see that f and g have a common fixed point. ///

Corollary 3.4. [1]. Let f and g be surjective continuous selfmaps of a complete metric space$(M,\,d)$. If there exists a real number $h>1$ such that

(4) $$d(fu,\,gv)\geq h\ \min\{d(u,fu),\ d(v,gv),\ d(u,v)\}$$

for each u,v in M, then f or g has a fixed point or f and g have a common fixed point.

Proof. Note that the metric d induces a mapping $\mathscr{F}:M\times M\rightarrow\mathscr{L}$ via $\mathscr{F}(u,v)=Fu,v$ $=H(x-d)(u,v))$ where $H(x)=0$ for $x\leq 0$ and $H(x)=1$ otherwise. If t-norm t is $t(a,b)=max(a,b)$ then clearly$(M,\mathscr{F},t)$ is a complete Menger space.

Let $d(a,b)=\min\{d(u,fu),\ d(v,gv),\ d(u,v)\}$.

Then $d(fu,gv)\geq hd(a,b)$ for $x\geq d(a,b)$ implies $F_{fu,gv}(hx)\leq F_{a,b}(x)$ and $F_{a,b}(x)=1$. Also, for $x<d(a,b)$

$$hx<hd(a,b)\leq d(fu,gv) \text{ gives } F_{fu,gv}(hx)=0.$$

Thus $F_{fu,gv}(hx)\leq F_{a,b}(x)$. So f, g satisfying (4) on (M, d) satisfy (3) on $(M, \mathscr{F}, t)$. The rest of the proof may be completed by Theoerm 3.3. ///

Corollary 3.5[1]. Let f,g be surjective selfmaps of a complete metric space(M,d). Suppose there exists a constant $h>1$ such that

$$d(fu, gv)\geq hd(u, v)$$

for each u,v in M. Then f and g have a unique common fixed point.

Remark 3.6. Taking $f=g$ in Corollaries 3.3 and 3.4 we get Theorems 1 and 3 respectively of [5].

Remark 3.7. As remarked by Rhoades [1], further generalizations of our results are not possible.

References

1. B.E. Rhoades, Some fixed point theorems for a palr of mappings, Jnānābha 15(1985), 151~156.
2. B. Schweizer and A. Sklar, Probabilistic Metric Spaces, North Holland Series, New York, 1983.
3. S.L. Singh and B.D. Pant, Common fixed point theorems in probabilistic metric spaces and extension to uniform spaces, *Honam Math. J.* **6**(1984), 1~12.
4. S.L. Singh, S.N. Mishra and B.D. Pant, General fixed point theorems in probabilistic metric and uniform spaces, *Indian J. Math.* **28**(1986).
5. S.Z. Wang, B. Yi, Z.M. Gao and K. Iséki, Some fixed point theorems on expansion mappings, *Math. Japon.* **29**(1984), 631~635.