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A Subclass of Starlike Functions

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Abstract

Let M be a positive real number and c a complex number such that $|c-1|<M\leq Re\{c\}$. Let $f, f(z)=z+a_2z^2+\ldots$, be analytic and univalent in the unit disc. It is said to belong to the class $S(c,M)$ if $|zf'(z)/f(z)-c|<M$. We find growth and rotation theorems for the class $S(c,M)$.

1. Introduction

Let S denote the class of functions $f(z)=z+a_2z^2+\ldots$, which are analytic and univalent in the open unit disc $E=\{z, |z|<1\}$. Let $0\leq\alpha<1$ and $S^*(\alpha)$ be the subclass of S which is starlike of order α. Let m and M be positive real numbers such that $|m-1|<M\leq m$. A function $f\varepsilon S$ is said to belong to the class $S(m,M)$ if, for z in E,

$$\left|\frac{zf'(z)}{f(z)}-m\right|<M.$$

It is clear that $S(m,M)\subset S^*(m-M)$. The class $S(m,M)$ was introduced by Jakubowski [2] and has also been studied in [1], [4] and [5]. In this paper we generalize the class $S(m,M)$ by letting m be complex. Let M be a positive real number and c be a complex number such that

$$|1-c|<M\leq Re\{c\}.$$

Let a function f of S belong to the class $S(c.M)$ if, for z in E,

$$\left|\frac{zf'(z)}{f(z)}-c\right|<M.$$

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We shall study growth and rotation theorems for the class $S(c, M)$.

2. A Growth Theorem for $S(c, M)$

Let $f \varepsilon S(c, M)$ and define

$$g(z) = [zf'(z)/f(z) - c]/M. \quad \cdots\cdots (1)$$

For $|z|<1$, the function g is analytic and bounded, $|g(z)|<1$, hence the values of $g(z)$ are contained in the disc C onto which $|z|<r$ is mapped by the linear transformation

$$w = \frac{g(o)+z}{1+\overline{g(o)}z} \quad \cdots\cdots (2)$$

([3], p. 167). It is convenient to define

$$b = g(o) = (1-c)/M, \quad \cdots\cdots (3)$$

and, if $c \neq 1$,

$$d = M(1-|b|^2)/|b|. \quad \cdots\cdots (4)$$

Suppose the disc C has centre at the point β and has radius ρ. Since the mapping (2) has the inverse

$$z = \frac{b-w}{-1+\bar{b}w},$$

the points $(b-\beta)/(-1+\bar{b}\beta)$ and $-1/\bar{b}$ are inverse with respect to the circle $|z|=r$, hence

$$\frac{b-\beta}{-1+\bar{b}\beta} = -\frac{k}{\bar{b}} \quad \cdots\cdots (5)$$

for some $k>0$. Also

$$r^2 = \left|\frac{b-\beta}{-1+\bar{b}\beta}\right| \left|\frac{-1}{\bar{b}}\right| = \frac{k}{|b|^2}.$$

Thus $k = |b|^2 r^2$ and (5) gives

$$\beta=\frac{b(1-r^2)}{1-|b|^2r^2}. \quad\cdots\cdots(6)$$

The points b and $1/\bar{b}$ are inverse with respect to the circle $|z-\beta|=\rho$, hence $|\beta-b|$ $|\beta-1/\bar{b}|=\rho^2$ which gives

$$\rho=\frac{(1-|b|^2)r}{1-|b|^2r^2} \quad\cdots\cdots(7)$$

We shall now prove the following

Theorem 1. Let $f\varepsilon S(c,M)$ and b and d as defined above, then for $|z|=r<1$,

$$re^{-Mr}<|f(z)|\leq re^{Mr}, \text{ (if } c=1),$$
$$r(1-|b|r)^{d\left(\frac{1}{2}+\frac{Re\{b\}}{2|b|}\right)}(1+|b|r)^{d\left(-\frac{1}{2}+\frac{Re\{b\}}{2|b|}\right)}\leq|f(z)|$$
$$\leq r(1+|b|r)^{d\left(\frac{1}{2}+\frac{Re\{b\}}{2|b|}\right)}(1-|b|r)^{d\left(-\frac{1}{2}+\frac{Re\{b\}}{2|b|}\right)} \text{ (if } c\neq 1).$$

Proof Let $g(z)$, β and ρ be as defined above. We have

$$Re\{\beta\}-\rho\leq Re\{g(z)\}\leq Re\{\beta\}+\rho \quad\cdots\cdots(8)$$

First let $c\neq 1$, $c=c_1+ic_2$ and $b=b_1+ib_2$. The inequality (8), in terms of $f(z)$, becomes

$$\frac{Mb_1(1-r^2)}{1-|b|^2r^2}-\frac{M(1-|b|^2)r}{1-|b|^2r^2}\leq Re\left\{\frac{zf'(z)}{f(z)}\right\}-c_1\leq\frac{Mb_1(1-r^2)}{1-|b|^2r^2}+\frac{M(1-|b|^2)}{1-|b|^2r^2}$$

A simple calculation gives

$$-\frac{M(1-|b|^2)(1+b_1r)r}{1-|b|^2r^2}\leq Re\left\{\frac{zf'(z)}{f(z)}-1\right\}\leq\frac{M(1-|b|^2)(1-b_1r)r}{1-|b|^2r^2}$$

Since $Re\{zf'(z)/f(z)-1\}=r\frac{\partial}{\partial r}\log|f(z)/z|$, the above inquality gives, on integration from 0 to r,

$$\frac{b_1d}{2|b|}\log(1-|b|^2r^2)-\frac{d}{2}\log\frac{1+|b|r}{1-|b|r}\leq\log\left|\frac{f(z)}{z}\right|$$
$$\leq\frac{b_1d}{2|b|}\log(1-|b|^2r^2)+\frac{d}{2}\log\frac{1+|b|r}{1-|b|r}, \quad\cdots\cdots(9)$$

where $d=M(1-|b|^2)/|b|$. On exponentiation we get the desired growth theorem for $S(c,M)$.

If $c=1$, then $b=0$ and we have

$$-Mr\leq Re\left\{\frac{zf'(z)}{f(z)}-1\right\}\leq Mr,$$

which leads to

$$r\ \exp(-Mr)\leq|f(z)|\leq r\ \exp(Mr).$$

The function $f(z)=z\ \exp(Mz)$ shows that the result is sharp.

This completes the proof of the theorem. ///

Remark If c is real we obtain the results of Silverman [4].

3. Rotation Theorems for $S(c, M)$

The function g as defined in (1) maps E onto a domain which is contained in a disc centered at β and has radius ρ. Hence the image of $zf'(z)/f(z)$ is contained in a disc C' centered at $M\beta+c$ and of radius $M\rho$. Also $M\rho<Re\{M\beta+c\}$, thus the origin lies exterior to C' and

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|\leq|\arg\{M\beta+C\}|+\text{arc sin}\frac{M\rho}{|M\beta+c|}\quad\cdots\cdots(10)$$

Since

$$M\beta+c=\frac{1-r^2+(1-|b|^2)cr^2}{1-|b|^2r^2},$$

(10) becomes

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|\leq\text{arc tan}\frac{(1-|b|^2)|Im\{c\}|r^2}{1-r^2+(1-|b|^2)\ Re\{c\}r^2}$$
$$+\text{arc sin}\frac{\rho}{|\beta+c/M|}.\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(11)$$

To consider sharpness of the above result, let

$$f_o(z)=z(1+\bar{b}z)^{M(1-|b|^2)/\bar{b}}.$$

We have

$$[zf'_o(z)/f_o(z)-c]/M=\frac{z+b}{1+\bar{b}z}$$

Since $|b|<1$, the right hand side maps the unit disc onto itself, hence $f_o\varepsilon S(c,M)$. Obviously f_o attains the bound in (11).

Now consider the imaginary part of $g(z)$. It is clear that

$$-\rho+\text{Im}\{\beta\}\leq\text{Im}\{g(z)\}\leq\rho+\text{Im}\{\beta\}$$

In terms of the function f, we get

$$-M\rho+M\ \text{Im}\{\beta\}+\text{Im}\{c\}\leq\text{Im}\left\{\frac{zf'(z)}{f(z)}\right\}=\text{Im}\left\{\frac{zf'(z)}{f(z)}-1\right\}\leq M\rho+M\ \text{Im}\{\beta\}+\text{Im}\,(c)$$

or

$$-\frac{(1-|b|^2)r(M-\text{Im}\{c\}r)}{1-|b|^2r^2}\leq\text{Im}\left\{\frac{zf'(z)}{f(z)}-1\right\}\leq\frac{(1-|b|^2)r(M+\text{Im}\{c\}r)}{1-|b|^2r^2}\cdots(12)$$

Since $r\dfrac{\partial}{\partial r}\arg\{f(z)/z\}=\text{Im}\{zf'(z)/f(z)-1\}$. an integration from 0 to r gives, if $b\neq0$,

$$-\frac{d}{2}\log\frac{1+|b|r}{1-|b|r}-\frac{d\ \text{Im}\{c\}}{2|b|M}\log\,(1-|b|^2r^2)\leq\arg\left\{\frac{f(z)}{z}\right\}$$
$$\leq\frac{d}{2}\log\frac{1+|b|r}{1-|b|r}-\frac{d\ \text{Im}\{c\}}{2|b|M}\log\,(1-|b|^2r^2)\cdots\cdots(13)$$

If $c=1$, *i.e.* $b=0$, then (12) gives

$$-Mr\leq\arg\left\{\frac{f(z)}{z}\right\}\leq Mr$$

The function $f(z)=ze^{Mz}$ shows that the result is sharp. We have not been able to establish the sharpness of (13). We combine the above results in the following theorem:

Theorem 2. Let $f\varepsilon S(c,M)$ and b,d,β and ρ as defined above. Then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|\leq\text{arc tan}\frac{(1-|b|^2)\,|\text{Im}\{c\}|\,r^2}{1-r^2+(1-|b|^2)\ Re\{c\}r^2}+\text{arc sin}\frac{\rho}{|\beta+c/M|}\cdots(14)$$

and

$$\left|\arg\left\{\frac{f(z)}{z}\right\}+\frac{d\ \text{Im}(c)}{2|b|M}\log(1-|b|^2r^2)\right|\leq\frac{d}{2}\log\frac{1+|b|r}{1-|b|r},\ \text{if }c\neq1\cdots(15\text{a})$$

$$\left|\arg\left\{\frac{f(z)}{z}\right\}\right| \leq Mr, \text{ if } c=1, \quad \cdots\cdots (15b)$$

The results (14) and (15b) are sharp.

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