HONAM MATHEMATICAL JOURNAL
Volume 9, Number 1. August 1987

On Geometric Properties of the Linear Invariant Families of Holomorphic Functions*

by
Bong-Goo Park
Honam College, Kwangju 500, Korea

1. Introduction

Let $\mathscr{A}(\mathscr{D})$ be the set of all analytic automorphisms of $\mathscr{D}$ onto $\mathscr{D}$ where $\mathscr{D}=\{z: |z|<1\}$. Pommerenk has defined [12] a family of functions of the form $f(z)=z+\ldots$, analytic and locally univalent ($f'(z)\neq 0$) in $\mathscr{D}$ to be a *linear invariant family* M if and only if for each $\varphi(z)$ in $\mathscr{A}(\mathscr{D})$ and every f in M the functional

$$\Lambda_\varphi[f(z)]=\frac{f[\varphi(z)]-f[\varphi(0)]}{f'[\varphi(0)]\varphi'(0)}=z+\ldots$$

is also in M. If M is a linear invariant family, then the order of M is defined as

$$\alpha=\sup\{|f''(0)/2|: f\in M\}.$$

Let U_α denote the union of all linear invariant families of order at most α. Then the universal family U_α is itself linear invariant. If $f(z)=z+\ldots$ is analytic and locally univalent in $\mathscr{D}$, then we may consider the linear invariant family M_f which it generates; namely

$$M_f=\{\Lambda_\varphi[f(z)]: \varphi(z)\in\mathscr{A}(\mathscr{D})\}.$$

The order of $f(z)$ is the order of the linear invariant family which it generates. As an aid in computing the order of $f(z)$, denoted ord(f), we have [12]

$$\begin{aligned}\mathrm{ord}(f)&=\sup_{z\in\mathscr{D}}|-\bar{z}+(1-|z|^2)f''(z)/2f'(z)|\\&=\sup\{|g''(0)/2|: g\in M_f\}.\end{aligned}$$

* This research is supported by the research grant of the Ministry of Education, 1986.
Received May 8, 1987. Accepted June 25, 1987.

In the present paper, we show that a linear invariant family M of order α generates the function

$$G(r)=\sup\{\max \arg f'(z):|z|=r,\ f\in M,\ 0\leq r<1\}$$

which is an increasing continuous function satisfying $G'(0^+)=2\alpha$ and $G(r)\geq 2$ arc sin r. Also we show that the associated function $T(t)=G(\tanh t)/2t$ satisfies $0\leq T(\infty)\leq T(t)\leq\alpha$ and $|\lambda|T_f(\infty)=T_{[\lambda f]}(\infty)$ where $[\lambda f]$ denotes the real scalar multiple of f in a vector space structure placed on locally schlicht functions due to Hornich.

Moreover Pommerenke's lower estimate on

$$\sup\{|\arg f'(z)|:|z|=r,\ f\in U_\alpha\}$$

is improved.

2. The Behaviour of arg $f'(z)$ for Linear Invariant Families

In order to gain some control over the behavior of arg $f'(z)$ for $f\in M$, we introduce the following: If M is a linear invariant family, we set

(2.1) $$G(r,M)=G(r)=\sup_{f\in M}\max_{|z|=r}\arg f'(z),\ 0\leq r<1,$$

where the argument varies continuously from the initial value of $\arg f'(0)=0$.

Lemma 2.1. For any linear invariant family M

$$G(r)=-\inf_{f\in M}\min_{|z|=r}\arg f'(z).$$

Proof. Let $f(z)$ be in M, z and ζ in $\mathcal{D}$ and

$$f(z,\zeta)=\frac{f\left(\dfrac{z+\zeta}{1+\zeta z}\right)-f(\zeta)}{f'(\zeta)(1-|\zeta|^2)}.$$

Since M is a linear invariant family, $f(z,\zeta)$ also belongs to M. If $z^*=(z+\zeta)/(1+\bar{\zeta}z)$ a brief calculation shows

(2.2) $$(1-|z|^2)f'(z,\zeta)=\frac{1-|z^*|^2}{1-|\zeta|^2}\cdot\frac{1+\bar{\zeta}z}{1+\zeta\bar{z}}\cdot\frac{f'(z^*)}{f'(\zeta)}$$

and, in particular, when $z=-\zeta$ we have

$$(2.3) \qquad (1-|z|^2)f'(-\zeta,\zeta)=1/(1-|\zeta|^2)f'(\zeta)$$

from which the Lemma 2.1 follows. ///

Since $\max_{|z|=r}$ arg $f'(z)$ is a monotone increasing function of r, $G(r)$ is also monotone increasing. In general, the supremum of monotone increasing piecewise analytic continuous functions need not be continuous, however we now show that $G(r)$ is in fact cotinuous.

Theorem 2.1. Let M be a linear invariant family of finite order. Let $\bar{M}$ denote the closure of M in the topology of uniform convergence on compacta. Let $M^*=\{f(sz)/s:\ f\in M \text{ and } 0<s\leq 1\}$. Then

$$(2.4) \qquad G(r,M)=G(r,M^*)=G(r,\bar{M}).$$

Furthermore $G(r)$ is a monotone increasing continuous function of r satisfying

$$G(r)\geq 2 \text{ arc sin } r,\ 0\leq r<1.$$

Proof. Since $G(r,M)\leq G(r,\bar{M})$ and $G(r,M)\leq G(r,M^*)$, to establish (2.4) we will show that $G(r,\bar{M})\leq G(r,M)$ and then that $G(r,M^*)\leq G(r,M)$. $\bar{M}$ is a compact linear invariant family, hence there is an $f(z)$ in $\bar{M}$ such that $G(r,\bar{M})=\arg f'(r)$. Since $\bar{M}$ is the closure of M, there is a sequence f_n in M which converges to f locally uniformly. Thus $G(r,M)\geq\lim_{n\to\infty}\arg f_n'(r)=G(r,\bar{M})$. To obtain the second inequality we choose a sequence f_n in M^* and a sequence z_n in $\mathscr{D}$, $|z_n|=r$, such that $\arg f_n'(z_n)\to G(r,M^*)$. Since $f_n'(z_n)=g_n'(s_nz_n)$, $g_n\in M, 0<s_n\leq 1$, we have

$$\arg f_n'(z_n)\leq\max_{|z|=r}\arg g_n'(z)\leq G(r,M).$$

Taking the limit as $n\to\infty$ yields $G(r,M^*)\leq G(r,M)$ and completes the proof of (2.4). Since $G(r)$ is monotone increasing, in order to establish the continuity of $G(r)$ it suffices to show $G(r^-)\geq G(r^+)$ for all r in $(0,1)$. We may assume M is compact by (2.4). Choose f_n in M and $r_n\to r$ such that $\arg f_n'(r_n)\to G(r^+)$. By compactness there is an f in M such that $\arg f'(r)=G(r^+)$.

The continuity of $\arg f'(r)$ implies

$$G(r^-)\geq \arg f'(r)=G(r^+).$$

If $f(z)$ is any function in M, then $g(z)=2f(z/2)$ is in M^* and satisfies

$$(1-z)g''(z)/g'(z)=(1-z)f''(z/2)/2f'(z/2).$$

Consequently, $\lim_{z\to 1}(1-z)g''(z)/g'(z)=0$ and, by Satz 3.14 in [13], the function $z/(1+z)$ is in $\bar{M}^*$. Since $G(r, M^*)=G(r,\bar{M}^*)$ and $\max_{|z|=r} \arg z/(1+z)=2 \arc\sin r$, we have

$$G(r)=G(r,M^*)\geq 2 \text{ arc } \sin r. \quad ///$$

Corollary 1. If M is a linear invariant family of finite order, then

$$\sup_{f\in M} \sup_{z\in \mathscr{D}} \arg f'(z)\geq \pi.$$

Corollary 2. If M is any linear invariant family of convex univalent functions, then $G(r)\equiv 2$ arc sin r.

Proof. This is immediate from Theorem 2.1 and the fact that $|\arg f'(z)|\leq 2$ arc sin r for any convex univalent function. ///

3. Rotationally Invariant Family

Kirwan [7] defines a family M to be *rotationally invariant* if whenever f is in M then $f(tz)/t$, $0<|t|\leq 1, t$ complex, is also in M. The convex functions, close-to-convex functions, V_k, S, and U_α [3] are examples of linear invariant families which are also rotationally invariant.

Theorem 3.1. If M is a compact rotationally linear invariant family of finite order, then M contains the function $z/(1+z)$.

Proof. If M is compact rotationally linear invariant, then $M=M^*=\bar{M}^*$ and the last part of the proof of Theorem 2.1 shows that $z/(1+z)$ must be in M. ///

For several well-known linear invariant families $G(r,M)$ can be determined explicitly. In addition to the convex functions which have already been considered in Corollary 2, close-to-convex functions have $G(r)=4$ arc sin r, functions in V_k have G

$(r)=k$ arc sin r, β-close-to-V_k functions satisfy $G(r)=(k+2\beta)$ arc sin r [2] and for the class $S, G(r)=4$ arc sin r, $0\leq r\leq 1/\sqrt{2}$ and $G(r)=\pi+\log[r^2/(1-r^2)]$, $1/\sqrt{2}\leq r<1$, [5]. It is evident from Theorem 2.1 that $G(r)$ does not determine the class M, however the following results show that $G(r)$ does uniquely define the order of M.

Theorem 3.2. Let M be a linear invariant family of order α, let $t\in(0,\infty)$, $r=\tanh t$, and define

$$(3.1)\qquad T(t)\equiv G(\tanh t)/2t=\sup_{f\in M}\max_{|z|=\tanh t}(1/2t)\arg f'(z).$$

Then 1) $T(t)=-\inf_{f\in M}\min_{|z|=\tanh t}(1/2t)\arg f'(z)$.

2) $(t_1+t_2)T(t_1+t_2)\leq t_1T(t_1)+t_2T(t_2)$.

3) $\lim_{t\to\infty}T(t)=T(\infty)$ exists.

4) $0\leq T(\infty)\leq T(t)\leq\alpha$.

5) $0\leq T(\infty)\leq(\alpha^2-1)^{1/2}$.

6) $T(t)$ is continuous in $(0,\infty)$ and $\lim_{t\to 0}T(t)=\alpha$.

7) For every $\alpha\geq 1$ there is a linear invariant family of order α with $T(\infty)=\gamma$ where γ is any number in $[0,(\alpha^2-1)^{1/2}]$.

Proof. The first claim follows directly from Lemma 2.1. Let $t_k(k=1,2)$ be given in $(0,\infty)$, $r_k=\tanh t_k$ and $z_k=r_ke^{i\theta}$. If $r=\tanh(t_1+t_2)$ and $z=re^{i\theta}$, then $(z_1+z_2)/(1+z_1\bar{z}_2)=z$. Using z, z_1, and z_2 in (2.2) yields

$$(1-|z_1|^2)f'(z_1,z_2)=\frac{1-|z|^2}{1-|z_2|^2}\cdot\frac{f'(z)}{f'(z_2)}$$

which implies

$$\begin{aligned}\arg f'(z)&=\arg f'(z_1,z_2)+\arg f'(z_2)\\&\leq 2t_1T(t_1)+2t_2T(t_2).\end{aligned}$$

Hence

$$2(t_1+t_2)T(t_1+t_2)\leq 2t_1T(t_1)+2t_2T(t_2)$$

which proves the second assertion.

The third claim follows immediately from 2) and a problem in Pólya and Szegö [11].

Furthermore, 2) implies $T(nt)\le T(t)$ for any integer n, thus $T(\infty)\le T(t)$ for all t in $(0,\infty)$. Since $r=\tanh t$ is equivalent to $t=(1/2)\log(1+r)/(1-r)$, the estimates [12]

$$|\log(1-|z|^2)f'(z)|\le\alpha\log(1+r)/(1-r)$$

and

$$|\arg f'(z)|\le(\alpha^2-1)^{1/2}\log(1+r)/(1-r)$$

immediately yield $T(t)\le\alpha$ and $T(\infty)\le(\alpha^2-1)^{1/2}$, which completes the proof of 4) and 5).

The first part of 6) follows from Theorem 2.1. Since $T(t)\le\alpha$, to prove the remainder of 6) it suffices to show $\lim_{t\to 0+}\inf T(t)\ge\alpha$. As in Theorem 2.1 we may assume that M is compact and choose an f in M such that $f''(0)/2=a_2=\alpha$.

Thus for z sufficiently small,

$$\arg f'(z)=\arg(1+2a_2z+0(z^2))$$

and

$$\max_{|z|=r}\arg f'(z)=\arc\sin[2a_2r+0(r^2)).$$

Consequently

$$\begin{aligned}T(t)&=\sup_{f\in M}\max_{|z|=\tanh t}\frac{\arg f'(z)}{2t}\\&\ge\frac{\arc\sin[2a_2r+0(r^2)]}{\log(1+r)/(1-r)}\end{aligned}$$

and

$$\lim_{t\to 0+}\inf T(t)\ge\lim_{r\to 0+}\frac{\arc\sin[2a_2r+0(r^2)]}{\log(1+r)/(1-r)}=a_2=\alpha.$$

Finally, let $\gamma\in[0,(\alpha^2-1)^{1/2}]$, $\alpha>1$, $C=\alpha(\alpha^2-1-r^2)^{1/2}(\alpha^2-1)^{-1/2}+ir$,

and

(3.2) $$f_c(z)=\frac{1}{2C}\left[\left(\frac{1+z}{1-z}\right)^c-1\right].$$

Then the order of $f(z)$ is $\{|C|^2+1+[(1-|C|^2)^2+4r^2]^{1/2}\}^{1/2}/\sqrt{2}$ which is α. Thus to prove 7) it suffices to show that $T(\infty)=\gamma$ for the linear invariant family M generated by $f_c(z)$.

For any $\varphi(z)$ in $\mathscr{A}(\mathscr{D})$ we have

$$\log \Lambda_\varphi'[f_c(z)]=\log[f_c'(\varphi(z))\varphi'(z)/f_c'(\varphi(0))\varphi'(0)]$$

and, letting $c=a+i\gamma$,

$$\arg \Lambda_\varphi'[f_c(z)]=\gamma \log\left|\frac{(1+\varphi(z))\ (1-\varphi(0))}{(1-\varphi(z))\ (1+\varphi(0))}\right|+(a+1)\arg\left(\frac{1+\varphi(z)}{1+\varphi(0)}\right)$$
$$+(a-1)\arg\left(\frac{1-\varphi(z)}{1-\varphi(0)}\right)+2\arg(1+\bar{\zeta}z).$$

If $z=re^{i\theta}$, then

$$|[1+\varphi(z)][1-\varphi(0)]/[1-\varphi(z)][1+\varphi(0)]|\leq(1+r)/(1-r)$$

and thus

$$\arg \Lambda_\varphi'[f_c(z)]\leq\gamma \log(1+r)/(1+r)+|a+1|\pi+|1-a|\pi+\pi \tag{3.3}$$
$$\leq\gamma \log(1+r)/(1-r)+3\pi,$$

where we have used the fact that $0\leq a\leq 1$.

On the other hand

$$\arg f_c'(r)=\gamma \log (1+r)/(1-r), \tag{3.4}$$

hence (3.3) and (3.4) yield

$$\gamma\leq G(r,M)/\log (1+r)/(1-r)=T(t)\leq\gamma+\frac{3\pi}{t}$$

which show that $T(\infty)=\gamma$ and completes the proof of the Theorem. ///

Corollary. Let M be a linear invariant family of order α. Then $G'(0^+)$ always exists and satisfies $G'(0^+)=2\alpha$.

Proof. We have

$$G'(0^+)=\lim_{r\to 0+}\frac{G(r)}{r}=\lim_{r\to 0+}\frac{\log(1+r)/(1-r)}{r}\ \frac{G(r)}{\log(1+r)/(1-r)}$$
$$=2\lim_{t\to 0+}T(t)=2\alpha.\ ///$$

4. Improvement of Pommerenke's Result

Pommerenke's best estimates [12] on $\arg f'(z)$ for $f(z)$ in U_α are $|\arg f'(z)|\leq 2\int_0^r \frac{(\alpha^2-x^2)^{1/2}}{1-x^2}dx\leq(\alpha^2-1)^{1/2}\log\frac{1+r}{1-r}+2\arc\sin r$ while, for any z in $\mathscr{D}$, there is an $f(z)$ in U_α with

$$(4.1)\qquad |\arg f'(z)|\geq(\alpha^2-1)^{1/2}\log(1+r)/(1-r).$$

One might therefore conjecture that for U_α, $G(r)$ is either $(\alpha^2-1)^{1/2}\log(1+r)/(1-r)$ or $(\alpha^2-1)^{1/2}\log(1+r)/(1-r)+2\arc\sin r$. Neither conjecture is true for any $\alpha>1$ since in the first case $G'(0)=2(\alpha^2-1)^{1/2}\neq 2\alpha$, while in the second $G'(0)=2[(\alpha^2-1)^{1/2}+1]\neq 2\alpha$. This suggests that it should be possible to improve (4.1) and it is as follows:

Theorem 4.1. For each α in $(1,\infty)$ and for each z satisfying $0<|z|<1/\alpha$, there is an $f(z)$ in U_α with $\arg f'(z)>(\alpha^2-1)^{1/2}\log(1+r)/(1-r)$.

Proof. Since U_α is rotationally invariant we may assume $z=r$, $0<r<1/\alpha$. Let

$$f_r(z)=\int_0^z(1+we^{i\lambda})^{\alpha-1}(1-we^{-i\lambda})^{-\alpha-1}dw$$

where $\lambda=\arc\cos r$. The function f_r is in $V_{2\alpha}$ since it is generated by the measure with weight $\alpha-1$ at $\theta=\lambda$ and weight $\alpha+1$ at $\theta=-\lambda$. Furthermore, $\arg f_n'(r)=2\alpha\arc\sin r$. Since $V_{2\alpha}\subset U_\alpha$, it now suffices to show $2\alpha\arc\sin r>(\alpha^2-1)^{1/2}\log(1+r)/(1-r)$ for $0<r<1/\alpha$. An elementary calculation shows that $h(r)=2\alpha\arc\sin r-(\alpha^2-1)^{1/2}\log(1+r)/(1-r)$ is a strictly increasing function of r, $r\in(0,1/\alpha)$, and, since $h(0)=0$, this completes the proof. ///

A careful examination of Pommerenke's proof that

$$|\arg f'(z)|\leq 2\int_0^r(\alpha^2-x^2)^{1/2}(1-x^2)^{-1}dx,\quad |z|=r,\quad f\in U_\alpha,$$

leads one to consider

$$f(z)=\int_0^z\exp[2i\int_0^w(\alpha^2-x^2)^{1/2}(1-x^2)^{-1}dx]dw$$

as a possible extremal function for the maximum of the argument of the derivative.

Indeed, in this case $\arg f'(r)=2\int_0^r(\alpha^2-x^2)^{1/2}(1-x^2)^{-1}dx$ which would certainly make it extremal. Unfortunately, $f(z)$ is not in U_α. This is difficult to verify directly from the definition of the order of $f(z)$, however if we note that $(1-z)f''(z)/f'(z)\to i(\alpha^2-1)^{1/2}$ as $z\to 1$ in any angle then f has as a limit function [13, Satz 3.14]

$$f_c(z)=(1/2c)\{[(1+z)/(1-z)]^c-1\},$$

where $c=-1+i(\alpha^2-1)^{1/2}$. Furthermore, the order of $f_c(z)$ is

$$\beta=[\alpha^2+1+(\alpha^4+2\alpha^2-3)^{1/2}]^{1/2}/\sqrt{2}$$

and a computation shows $\beta>\alpha$ for all $\alpha>1$. If M is the linear invariant family generated by $f(z)$, then $f_c(z)$ is in $\bar{M}$ and, since order M=order $\bar{M}$, it follows that order $f(z)\geq$ order $f_c(z)=\beta>\alpha$, which shows that $f(z)$ is not in U_α.

One fruitful method of investigation of U_α has been to place various normed linear space structures on $X=\bigcup_{\alpha\geq 1}U_\alpha$.

Following Hornich, we define

$$[f+g](z)=\int_0^z f'(w)g'(w)dw \qquad (f,\ g\in X),$$
$$[af](z)=\int_0^z[f'(w)]^a dw \qquad (f\in X,\ a \text{ real})$$

where square brackets denote the algebraic operations on X.

Theorem 4.2. If f is in X and a is real, then

$$|a|T_f(\infty)=T_{(af)}(\infty), \tag{4.2}$$

where $T_g(\infty)$ denotes the value of $T(\infty)$ for the linear invariant family M_g which g generates.

Proof. We actually show that

$$||a|T_f(t)-T_{(af)}(t)|\leq\pi|a-1|/2t, \tag{4.3}$$

from which (4.2) follows obviously. For any $(\varphi)z$ in $\mathscr{A}(\mathscr{D})$ and any r in $[0,1]$ a computation shows

$$|a|\,|\arg \Lambda_\varphi'(f(z))| = |\arg \Lambda_\varphi'([af](z)) + (a-1)\arg \varphi'(z)/\varphi'(0)|$$
$$\leq |\arg \Lambda_\varphi'([af](z))| + |a-1|\pi.$$

Therefore

$$|a|\,|\arg \Lambda_\varphi'(f(z))| \leq G(r, M_{(af)}) + |a-1|\pi$$

and consequently

$$|a| G(r, M_f) \leq G(r, M_{(af)}) + |a-1|\pi.$$

Upon reversing the roles of f and $[af]$, we obtain

$$\big|\,|a| G(r, M_f) - G(r, M_{(af)})\big| \leq |a-1|\pi$$

from which (4.3) follows directly. ///

Remark 4.1. It is perhaps appropriate to remark at this stage of development that a function $\beta(t)$, similar to $T(t)$, was introduced by Pommerenke for the study of the distortion of $|f'(z)|$ in linear invariant families. To his conclutions [12, Satz 2.2] one can add the facts that $\beta(t)$ is continuous, $\lim_{t\to 0^+} \beta(t)=\alpha$ and for each β in $[1, \alpha]$ there is a linear invariant family M with $\beta(\infty)=\beta$. There are several differences in the behaviour of $\beta(t)$ and $T(t)$. Although $\beta(t)$ may be constant, $T(t)$ can not. This is obvious since $T(\infty)<T(0)$. The function $T_{(af)}(\infty)$ is linear while $\beta_{(af)}(\infty)=1+(\beta_f(\infty)-1)|a|$. It would also be of interest to know if there is a proposition for $T(t)$ comparable to the following, due to Pommerenke; $\beta(\infty)=\alpha$ for a compact family M if and only if

$$(1/2\alpha)\{[(1+z)/(1-z)]^\alpha - 1\}$$

is in M.

Recall that a function f in X is said to have *boundary rotation* $k\pi$ if for $z=re^{i\theta}$

$$\sup_{r\in(0,1)} \int_0^{2\pi} |\mathrm{Re}\{1+zf''(z)/f'(z)\}|\, d\theta = k\pi. \tag{4.4}$$

$G(r)$ certainly does not depend on the boundary rotation of f. For example, we can find a function $f(z)$ in S with boundary rotation 100π. However, although f is in V_{100} but not $V_{k'}$ for any $k'<100$, it is not true that $G(r, M_f)=100$ arc sin r since any function in S satisfies $|\arg f'(z)|\leq 4$ arc sin r for $0\leq r\leq 1/\sqrt{2}$ [5].

We will now give a second illustration of the fact that $G(r)$ is not a function of the boundary rotation, this time concentrating on what happens for r close to 1.

Recall that Corollary 2 asserted that $G(1)/\pi=k/2$ for the case of convex univalent functions ($k=2$), where

$$G(1)=\lim_{r\to 1} G(r)=\sup_{f\in M}\sup_{|z|<1}\arg f'(z).$$

One might hope that this is indeed a phenomenon of boundary rotation and will persist for other values of k. To see that this is not the case consider a close-to-convex function $f(z)$ which has infinite boundary rotation. Then $f_s(z)=f(sz)/s$, $0<s<1$, satisfies $G(1)\leq 4\pi$ since $f_s(z)$ is also close-to-convex. On the other hand as $s\to 1$ the boundary rotation of $f_s\to\infty$ and we cannot have $G(1)/\pi=k/2$.

We can obtain a relationship between $G(1)$ and the order of M by utilizing the class $K(\beta)$ of generalized close-to-convex functions of order β. A function f in X is in $K(\beta)$, $\beta\geq 0$ if for each r in [0,1) and each pair θ_1 and θ_2, $0\leq\theta_1<\theta_2\leq 2\pi$, we have

$$\int_{\theta_2}^{\theta_1}\mathrm{Re}[1+re^{i\theta}f''(re^{i\theta})/f'(re^{i\theta})]d\theta=-\beta\pi; \tag{4.5}$$

equivalently if there is a $c\neq 0$ and a normalized convex univalent function $\varphi(z)$ such that for z in $\mathscr{D}$

$$|\arg\{cf'(z)/\varphi'(z)\}|\leq\beta\pi/2.$$

Theorem 4.3. Let M be a linear invariant family satisfying

$$\sup_{f\in M}\sup_{|z|<1}\arg f'(z)=\beta\pi<\infty.$$

Then $M\subset K(\beta)$, $|\arg f'(z)|\leq 2(\beta+1)$ arc sin r, and M is of finite order α, $\beta+1\geq\alpha\geq 1$.

Proof. We show that M is in $K(\beta)$ but not in $K(\beta-2)$ (when $\beta>2$). Let $z_2=e^{i\theta}z_1$, $|z_1|=r$, θ in $(0, 2\pi)$. Then for $z=re^{i\theta}$

$$\int_{\theta_2}^{\theta_1}Re[1+zf''(z)/f'(z)]d\theta=\arg[z_2f'(z_2)/z_1f'(z_1)].$$

We therefore set

$$\Psi(r,\theta)=\inf\{\arg[z_2 f'(z_2)/z_1 f'(z_1)]:\ f\in M\}.$$

Since

$$\arg\left[\frac{z_2 f'(z_2)}{z_1 f'(z_1)}\right]=\arg\left[\frac{z_2}{z_1}\left(\frac{1-|z_1|^2}{1-\bar{z}_1 z_2}\right)^2\right]+\arg f'(\zeta_0, z_1)$$

where $\zeta_0=(z_2-z_1)/(1-\bar{z}_1 z_2)$, we see that

(4.6) $$\Psi(r,\theta)=2 \operatorname{arc} \cot[(1-r^2) \cot (\theta/2)/(1+r^2)]+\inf_{f\in M} \arg f'(\zeta_0, z).$$

Because of the linear invariance of M,

$$\inf[\arg f'(\zeta_0, z):\ f\in M]=\inf[\arg f'(\zeta_0):\ f\in M]$$

and hence by the hypothesis we have

$$(2-\beta)\pi\geq\inf_{|z|<1} \Psi(r,\theta)\geq-\beta\pi.$$

Thus $M\subset K(\beta)$ but, if $\beta>2$, M is not in $K(\beta-2)$. The remainder of the theorem now follows from well-known results for $K(\beta)$. Namely, order $K(\beta)=\beta+1$ and $|\arg f'(z)|\leq 2(\beta+1) \operatorname{arc} \sin |z|$, f in $K(\beta)$. ///

Remark 4.2. Finally we remark that Theorem 3.2 may be used to show that certain families are not linear invariant. For example, as one type of generalization of V_k, by Pinchuk [1], let V_k^λ denote the class of functions in X which satisfy

$$\sup_{0\leq r<1}\int_0^{2\pi}|Re[e^{i\lambda}(1+zf''(z)/f'(z))]|\,d\theta=k\pi \cos \lambda,\ k\geq 2,\ |\lambda|<\pi/2.$$

One can show that

$$\alpha=\sup\{a_2:\ f\in V_k^\lambda\}=k|1+e^{-2i\lambda}|/4$$

while $T(\infty, V_k^\lambda)\geq(k+2)|\sin 2\lambda|/4$. It is easy to see that the inequality $T(\infty)\leq(\alpha^2-1)^{1/2}$ is not valid for various values of λ and k and hence, by Theorem 3.2, can not be a linear invariant family for those values.

References

1. D.A. Brannan, *On functions of bounded boundary rotation* I, Proc. Edinburgh Math. Soc., 2, 16 (1968~69), 339~347.
2. D.M. Campbell, *β-close-to-linear invariant families*, Ph.D. thesis, University of North Carolina at Chapel Hill, 1971.
3. D.M. Campbell, *Locally univalent functions with locally univalent derivatives*, Trans. Amer. Math. Soc., 162 (1971), 395~409.
4. H.B. Coonce and M.R. Ziegler, *The radius of close-to-convexity of functions of bounded boundary rotation*, Proc. Amer. Math. Soc., 35 (1972), 207~210.
5. G.M. Goluzin, *Geometric theory of functions of a complex variable*, *Vol.26*, Amer. Math. Soc., Providence, R.I., (1969).
6. W. Kaplan, *Close-to-convex schlicht functions*, Mich. Math. J., 1 (1952), 169~185.
7. W.E. Kirwan, *A note on extremal problems for certain classes of analytic functions*, Proc. Amer. Math. Soc., 17. (1966), 1028~1030.
8. O. Lehto and K.J. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. 97 (1957), 47~65,
9. Z. Nehari, *Conformal Mapping*, Dover Pub. Inc., New York, 1952.
10. J. Noonan, *Asymptotic behaviour of functions with bounded boundary rotation*, Trans. Amer. Math. Soc., 164 (1972), 397~410.
11. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, *Vol.1*, Berlin, 1925.
12. Ch. Pommerenke, *Linear-invariant Familien analytischer Funktionen* I, Math. Annalen, 155 (1964), 108~154.
13. Ch. Pommerenke, *Linear-invariant Familien analytischer Funktionen* II, Math. Annalen, 156 (1964), 226~262.