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Remarks on Coherent Analytic Sheaves

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1. Introduction

The contents of this thesis is described as follows. In section 2 we develop the general theory of sheaves used in later section. The main contents of §3 is to prove Theorem 3.6 concerning coherent analytic sheaves: Let $\mathscr{F}$ be a sheaf on a closed complex submanifold N of M, and $\tilde{\mathscr{F}}$ be the trivial extension of $\mathscr{F}$ to M. Then $\mathscr{F}$ is coherent if and only if $\tilde{\mathscr{F}}$ is coherent. Section 4 deals with the properties of coherent analytic sheaves. Especially, we concentrate the Theorem 4.2 which says that if $\mathscr{F}$ is coherent on M, then Supp $(\mathscr{F})$ is an analytic subset of M. Also using this Theorem we shall prove Corollary 4.3 that is one of local properties of coherent sheaves.

2. Preliminaries

Let $\mathscr{F}$ be a presheaf of rings on X. It is clear that $\{\mathscr{F}(U), \rho_{VU}\}$ is an *inductive system* (direct system). Thus, for each $x \in X$ the inductive limit ring is defined as

$$\mathscr{F}_x = \varinjlim_{x \in U} \mathscr{F}(U) \quad (U\text{: open in } X.) \qquad ([8]).$$

Each $\mathscr{F}_x$ is called the *stalhs* of $\mathscr{F}$ at x, and each element of $\mathscr{F}_x$ is called a *germ* at x. In this case, there is a natural mapping

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \mathscr{F}_x \\ \Cup & & \Cup \\ f & \longrightarrow & f_x \end{array}$$

and f_x is said to be the *germ* of f, where $U \subset X$ is an open set containing x.

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With the above notations, set

$$\widetilde{\mathscr{F}}=\bigcup_{x\in X}\mathscr{F}_x$$

and let $\pi:\widetilde{\mathscr{F}}\longrightarrow X$ be defined by $\pi^{-1}(x)=\mathscr{F}_x$ for all $x\in X$.

Given $f\in\mathscr{F}(U)$ (U: open in X), define

$$\begin{array}{rccc} \tilde{f}: & U & \longrightarrow & \widetilde{\mathscr{F}} \\ & \Cup & & \Cup \\ & x & \longrightarrow & \tilde{f}(x)=f_x. \end{array}$$

For a base of open sets for the topology of $\widetilde{\mathscr{F}}$, we take the family of sets $\tilde{f}(U)$ over all $U\in\mathscr{U}(x)$ and $f\in\mathscr{F}(U)$. It is obvious that the *local section* $\tilde{f}:U\longrightarrow\widetilde{\mathscr{F}}$, $f\in\mathscr{F}(U)$, are continuous in this topology. Moreover, it follows that

(i) $\pi:\widetilde{\mathscr{F}}\longrightarrow X$ is a local homeomorphism.

(ii) the induced topology on $\mathscr{F}_x\subset\widetilde{\mathscr{F}}$ is discrete for all $x\in X$.

([3]). We call the topological space $\widetilde{\mathscr{F}}$, together with the projection $\pi:\widetilde{\mathscr{F}}\longrightarrow X$, *sheafification* of the presheaf $\mathscr{F}$. In general, a sheaf is defined as follows.

Definition 2.1. Let $\mathscr{F}$ and X be topological spaces, and let $\pi:\mathscr{F}\longrightarrow X$ be a local homeomorphism. $(\mathscr{F},\pi,X)$ is called a *sheaf of rings* on X it

(i) each stalk $\mathscr{F}_x(=\pi^{-1}(x))$ has the structure of a ring for all $x\in X$.

(ii) the ring operations are continuous in the topology on $\mathscr{F}$.

It is clear that $(\widetilde{\mathscr{F}},\pi,X)$ as above is also a sheaf of rings on X.

For a complex manifold M with dimension $m\geq 0$, we define the contravariant functor

$$\begin{array}{rccl} \mathscr{O}_M: & \mathscr{U}(M) & \longrightarrow & Ring \\ & \Cup & & \Cup \\ & U & \longrightarrow & \mathscr{O}_M(U)=\{f:U\longrightarrow C\,|\,f \text{ is holomorphic}\}, \end{array}$$

where Ring is the category consisting of all rings and ring homomorphisms.

Of course, the functor $\mathscr{O}_M$ is a presheaf of rings on M. Let us denote the sheafification of $\mathscr{O}_M$ by the same letter $\mathscr{O}_M$. That is, we put $\mathscr{O}_M$ as a set

$$\mathscr{O}_M=\bigcup_{x\in M}\mathscr{O}_{M,x}$$

This sheaf $\mathcal{O}_M$ of rings on M is called the *OKA sheaf* of M or the *structure sheaf* of M ([10]). We shall use the following notations.

$$\mathcal{O}_M=\mathcal{O},\ \mathcal{O}_{M,x}=\mathcal{O}_x,\ \text{and}\ \mathcal{O}|_U=\mathcal{O}_U\ \text{(restriction to } U).$$

It is important to note that for each $x\in M, \mathcal{O}_x$ is a Noetherian local domain with unique factorization property. Furthermore, $f_x\in\mathcal{O}_x$ is a unit if and only if $f(x)\neq 0$ ([3], [4], [14]). Throughout this paper, $\mathcal{M}_{\mathcal{O}_x}$ will denote the maximal ideal of $\mathcal{O}_x$.

Let X be a topological space and $\mathcal{R}$ (resp, $\mathcal{F}$) be a sheaf of rings (resp, abelian groups) on X. Then $\mathcal{F}$ is said to be a *sheaf of $\mathcal{R}$-modules* on X or *$\mathcal{R}$-sheaf* on X if

(i) $\mathcal{F}_x$ has the structure of an $\mathcal{R}_x$-module for all $x\in X$

(ii) the module operations are continuous.

In particular, if $\mathcal{R}=\mathcal{O}_M$ then $\mathcal{F}$ is called an ***analytic sheaf*** on M. Let $\mathcal{F}$ be an analytic sheaf on M. Then we note that for each open subset U of M,

$$\mathcal{O}(U)=\{\sigma: U\longrightarrow\mathcal{O}\,|\,\sigma \text{ is a continuous section on } U\}$$

and

$$\mathcal{F}(U)=\{\sigma: U\longrightarrow\mathcal{F}\,|\,\sigma \text{ is a continuous section on } U\}.$$

Let M and N be complex manifolds. By a ***map*** $\phi: M\longrightarrow N$ we mean a holomorphic map of M to N ([7], [14], [15]). By the natural way the sheaf $\mathcal{O}_M$ is a sheaf of $\mathcal{O}_N$-module. That is, for each $x\in M$ and each $g_{\phi(x)}\in\mathcal{O}_{N,\phi(x)}$

$$g_{\phi(x)}\cdot\mathcal{O}_{M,x}=(g\circ\phi)_x\cdot\mathcal{O}_{M,x}\cdots\cdots\cdots\cdots\cdots\text{(A)}$$

Definition 2.2. Let Y be a subset of a complex manifold M. Then Y is said to be an ***analytic subset*** of M if for each $x\in M$ there exists an open neighborhood U of x and a holomorphic function

$$f: U\longrightarrow C^p \text{ such that } Y\cap U=f^{-1}(0)\ (p \text{ may depend on } x)\ ([5],[9],[13]).$$

Notice that an analytic subset of M is necessarily closed.

Let Z be an analytic subset of M. For each open subset U of M, we put $I_Z(U)=\{f: U\longrightarrow C\,|\,f$ is holomorphic and $f|_Z\equiv 0\}$. Then $I_Z=\{I_Z(U),\rho_{VU}\}$ is a presheaf of ideals of $\mathcal{O}_M$. The sheafification of I_Z is denoted by the same letter I_Z. Then for each $x\in M$, we have

$$I_{Z,x}=\begin{cases}\mathcal{O}_{M,x} & \text{if } x\notin Z\\ \text{an ideal of } \mathcal{O}_{M,x} & \text{if } x\in Z.\end{cases}$$

We put the quotient sheaf $\mathcal{O}_M/I_Z=\mathcal{O}_Z$, then we have an exact sequence

$$0\longrightarrow I_Z\longrightarrow \mathcal{O}_M\longrightarrow \mathcal{O}_Z\longrightarrow 0 \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(\text{B}).$$

Definition 2.3. For complex manifolds M and N, let ϕ: $M\longrightarrow N$ be a map (i.e., holomorphic map).

(i) <Inverse image sheaves>

Let $\mathcal{F}$ be an analytic sheaf on N i.e., $\mathcal{F}=(\mathcal{F},\pi,N)$.
We put

$$\phi^{-1}\mathcal{F}=\bigcup_{x\in M}\mathcal{F}_{\phi(x)}$$

For each open subset U of N, we can choose an open subset V of M with $\phi(V)\subset U$. Take an element $s\in\mathcal{F}(U)$, we put

$$\tilde{S}(U)=\{s_{\phi(x)}\mid x\in V\}.$$

For a basis of open sets with respect to the topology on $\phi^{-1}\mathcal{F}$, we take the family of sets $\tilde{S}(U)$. Then for each $x\in M$, $(\phi^{-1}\mathcal{F})_x$ is an $\mathcal{O}_{N,\phi(x)}$-module. It follows from (A) that $\mathcal{O}_{M,x}$ is an $\mathcal{O}_{N,\phi(x)}$-module, and we also can define

$$\phi^*\mathcal{F}=\phi^{-1}\mathcal{F}\otimes_{\phi^{-1}\mathcal{O}_N}\mathcal{O}_M$$

Then $\phi^*\mathcal{F}$ is an analytic sheaf on M which is called the *analytic inverse image sheaf* of $\mathcal{F}$. Note that, for each $x\in M$,

$$(\phi^{-1}\mathcal{F})_x=\mathcal{F}_{\phi(x)}$$
$$(\phi^*\mathcal{F})_x=\mathcal{F}_{\phi(x)}\otimes_{\mathcal{O}_{N,\phi(x)}}\mathcal{O}_{M,x}$$

(ii) <Direct image sheaves>

Let $\mathcal{F}$ be an analytic sheaf on M. For each open subset U of N, $\{\mathcal{F}(U),\ \rho_{VN}\}$ is a presheaf. The sheafification of this presheaf, denoted by $\phi_*\mathcal{F}$, is called the *direct imagesheaf* of $\mathcal{F}$. Since $(\phi_*\mathcal{F})_{\phi(x)}\subset\Pi\mathcal{F}_x$, $\phi_*\mathcal{F}$ is a sheaf of $\mathcal{O}_M$-modules. For each $x\in M$, since $1\in\mathcal{O}_{M},x$ and $\mathcal{O}_M$ is a sheaf of $\mathcal{O}_N$-modules, $\phi_*\mathcal{F}$ is an analytic sheaf on N.

The following are elementary properties related to ϕ_* and ϕ^* whose proofs can be referred in ([2], [3], [4], [8]).

(a). ϕ_* a is a left exact functor.
(b). ϕ^* isa right exact functor.

Let X be a topological space and Z a closed subspace of X. For the inclusion map $i: Z \longrightarrow X$ and a sheaf $\mathscr{F}$ on Z, $i_*\mathscr{F}=\widetilde{\mathscr{F}}$ is called the *trivial extension* of $\mathscr{F}$ to X, or the *sheaf of X obtained by extending $\mathscr{F}$ by zero outside Z.*

Proposition 2.4. With the notations above the following hold:

(i) $$\widetilde{\mathscr{F}}_X=\begin{cases} 0 & \text{if } x\notin Z \\ \mathscr{F}_x & \text{if } x\in Z. \end{cases}$$

(ii) $$i^{-1}\widetilde{\mathscr{F}}=\mathscr{F}.$$

3. Coherent Sheaves

In this section, by X(resp, M) we denote a topological space (resp, complex manifold).

Let $\mathscr{F}$ be an $\mathscr{R}$-sheaf on X. Then finitely many sections $s_1, \cdots, s_p \in \mathscr{F}(U)$ define an $\mathscr{R}_U$-homomorphism

$$\sigma: \mathscr{R}_U{}^p \longrightarrow \mathscr{F}, \quad (a_{1,x}, \cdots, a_{p,x}) \longrightarrow \sum_1^p a_{i,x}s_{i,x}(x\in U).$$

We say that $\mathscr{F}_U$ is *generated by the sections* $s_1, \cdots, s_p$ if σ is surjective. An $\mathscr{R}$-sheaf $\mathscr{F}$ is called *finitely generated* or *of finite type* at $x\in X$ if there is an open neighborhood U of x such that $\mathscr{F}_U$ is generated by finitely many sections in $\mathscr{F}$ over U.

Definition 3.1. An $\mathscr{R}$-sheaf $\mathscr{F}$ is called *of finite type* on X if it is of finite type at all points $x\in X$.

Let $\mathscr{F}$ be an $\mathscr{R}$-sheaf on X. If $\sigma: \mathscr{R}_U{}^p \longrightarrow \mathscr{F}_U$ is an $\mathscr{R}_U$-homomorphism determined by sections $s_1, \cdots, s_p \in \mathscr{F}(U)$, the *sheaf of relations* of $s_1, \cdots, s_p$ is defined by

$$\mathscr{R}el(s_1, \cdots, s_p)=\mathrm{Ker}(\sigma)=\bigcup_{x\in U}\{(a_{1,x}, \cdots, a_{p,x})\in \mathscr{R}_x{}^p \mid \sum_1^p a_{i,x}s_{i,x}=0\}.$$

Obviously this is an $\mathscr{R}_U$-submodule of $\mathscr{R}_U{}^p$. An $\mathscr{R}$-sheaf $\mathscr{F}$ is called *of relation finite type* at $x\in X$ if, for every finite system $s_1, \cdots, s_p$ of sections over an open neighborhood U of x, the sheaf of relations $\mathscr{R}el\ (s_1, \cdots, s_p)$ is of finite type at x.

Definition 3.2. An $\mathscr{R}$-sheaf $\mathscr{F}$ is called *of relation finite type* on X if $\mathscr{F}$ is of relation finite type at all points of X.

The following property can be proved by using the Weierstrass Preparation Theorem ([3], [6], [11], [12]).

Property 1. For an open subset U of M and $s_1, \cdots, s_p \in \mathcal{O}_M{}^q(U)$, the sheaf of relations $\mathscr{R}el(s_1, \cdots, s_p)$ is a subsheaf of $\mathcal{O}_U{}^p$ of finite type.

Definition 3.3. An $\mathscr{R}$-sheaf $\mathscr{F}$ on X is called *coherent* if $\mathscr{F}$ is of finite and of relation finite type on X.

An analytic sheaf $\mathscr{F}$ on M is said to be *of finite representation* if, for each $x \in M$, there exist an open neighborhood U of x such that the sequence

$$\mathcal{O}_U{}^p \longrightarrow \mathcal{O}_U{}^q \longrightarrow \mathscr{F}_U \longrightarrow 0$$

is exact.

Property 2. Every analytic subsheaf of $\mathcal{O}_M{}^q$ which is of finite type is coherent.

By Definition 3.3. and Property 2, the following is easily proved.

Property 3. Let $\mathscr{F}$ be a coherent analytic sheaf on M. Then for any open subset U of M and $s_1, \cdots, s_p \in \mathscr{F}(U)$, $\mathscr{R}el(s_1, \cdots, s_p)$ is also a coherent sheaf.

Let dim $M = m$. Then from the Hilbert Syzygy Theorem ([3], [4]). and Definition 3.3, for every coherent analytic sheaf $\mathscr{F}$ on M we can find a free resolution of $\mathscr{F}$ over some open neighborhood U of each $x \in M$ such that

$$0 \longrightarrow \mathcal{O}_U{}^{p_m} \longrightarrow \mathcal{O}_U{}^{p_{m-1}} \longrightarrow \cdots\cdots \longrightarrow \mathcal{O}_U{}^{p_0} \longrightarrow \mathscr{F}_U \longrightarrow 0$$

([3]).

We have the following basic properties of coherent analytic sheaves whose proofs are referred in ([3], [4]).

Property 4. (i). Suppose $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{K} \longrightarrow 0$ is a short exact sequence of analytic sheaves on M. If any two of the sheaves $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{K}$ are coherent, so is third.

(ii). Let α: $\mathscr{F} \longrightarrow \mathscr{G}$ be a homomorphism of coherent analytic sheaves on M. Then Ker(α), Im(α) and Coker(α) are also coherent analytic sheaves on M.

Proposition 3.4. Let N be a complex submanifold of M. Then I_N and $\mathcal{O}_N = \mathcal{O}_M / I_N$ are coherent analytic sheaves on M.

Proof. (a). If N is an open submanifold of M, then, by the uniqueness of analytic continuation (Note dim $_cM=$dim $_cN$), $I_N=0$ and so I_N is trivially coherent.

(b). In case of $\dim_c N=k<m(=\dim {}_cM)$. Let $\{(U,\phi_U)\}$ be a local coordinate system of M. Then since $\phi_U:U\approx C^m$, we may assume that $M=C^m$ and $N=C^k=\{(z_1,\cdots,z_k,0,\cdots,0)\in C^m\}$. By our definition, we have $I_N(U)=\{f\in\mathcal{O}_M(U)\,|\,f|_N=0\}$ for each open subset $U\subset C^m$. But since $U\cap N=\phi$ implies $I_N(U)=\mathcal{O}_M(U)$, $I_{N,z}=\mathcal{O}_{M,z}$ for all $z\notin N$. Hence I_N is coherent on $M-N$, because $\mathcal{O}_M$ is coherent.

If $z\in N$ and U is an open neighborhood of z, then $f(z_1,\cdots,z_k,0,\cdots,0)\equiv 0$ for each $f\in I_N(U)$ where $(z_1,\cdots\cdots,z_k,0,\cdots,0)\in N\cap U$. Writting z_i as i-th coordinate function for $i=1,2,\cdots,m$, f is gener ated by $\{z_{k+1},\cdots,z_m\}$. Thus I_N is of finite type and so I_N is coherent on N (by Property 2).

And also since I_N and $\mathcal{O}_M$ are coherent sheaves, by (ii) of Porperty 4 $\mathcal{O}_N$ is a coherent sheaf. *Q.E.D*

Let Y be a proper analytic subset of M. If for each $x\in Y$ there exist an open neighborhood U of x in M and an element $f\in A(U)$ such that $f^{-1}(0)=U\cap Y$, then we say that Y is an *analytic hypersurface* of M.

It is remarked that Y is an analytic hypersurface if and only if $I_{Y,x}$ is a principal ideal of $\mathcal{O}_{M,x}$ for each $x\in Y$ (Vol. 1 of [3]) (C).

Proposition 3.5. Let Y be an analytic hypersurface of M. Then I_Y and $\mathcal{O}_Y$ are coherent sheaves.

Proof. As in the proof of proposition 3.4, in our assertion being local we may assume that $M=C^m$ and Y is an analytic hypersurface of C^m. By (C) above, for each $x\in Y$, since $I_{Y,x}$ is a principal ideal of $\mathcal{O}_{M,x}$, we can put $I_{Y,x}=(P_x)$, *i.e.*, P_x is a generator of $I_{Y,x}$. Let P be a represenitive for P_x. If, for $f\in A(U)$, $f^{-1}(0)=U\ Y$ contains x, then there exists an open neighborhood V of x in U such that $I_{Y,y}=(P_y)$ for all $y\in V\cap Y$ (see Vol. 1 of [3]). That is, I_Y is of finite thpe. By Property 2, I_Y is a coherent sheaf. Moreover, by (ii) of Property 4, $\mathcal{O}_Y$ is of course a coherent sheaf. ///

Let N be another complex manifold. A holomorphic map ϕ: $M\longrightarrow N$ is said to be *finite* if ϕ is a closed and $\phi^{-1}(\phi(x))$ is a finite set for each $x\in M$. Therefore, if M is a closed submanifold of N and ϕ is the inclusion map, then ϕ is a finite map. Furthermormore, if a holomorphic map ϕ: $M\longrightarrow N$ is finite and the sequence of sheaves on M.

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is exact, then

$$0 \longrightarrow \phi_* \mathscr{F}' \longrightarrow \phi_* \mathscr{F} \longrightarrow \phi_* \mathscr{F}'' \longrightarrow 0$$

is the exact sequence of sheaves on N ([6]).

Property 5. Let ϕ: $M \rightarrow N$ be a holomorphic map and $x \in M$ be an isolated point of the fiber $\phi^{-1}(\phi(x))$. Then, there exist open neighborhoods U of x in M and V of $\phi(x)$ in N such that

(i) $\phi(U) \subset V$

(ii) the induced map ϕ_{UV}: $U \longrightarrow V$ by ϕ is finite.

(iii) for any coherent analytic sheaf $\mathscr{F}$ on U, the direct image sheaf $(\phi_{UV*})\ \mathscr{F}$ is a coherent analytic sheaf on V ([4], [6]).

Theorem 3.6. Let N be a colsed complex submanifold of M and ϕ: $N \longrightarrow M$ be the inclusion map. For a sheaf $\mathscr{F}$ on N, letting $\phi^* \mathscr{F} = \widetilde{\mathscr{F}}$ be the trivial extension of $\mathscr{F}$ to M. Then $\mathscr{F}$ is a coherent analytic sheaf on N if and only if $\widetilde{\mathscr{F}}$ is a coherent analytic sheaf on M.

Proof. ($\Rightarrow$). Clearly, $\widetilde{\mathscr{F}}$ has the structure of an $\mathcal{O}_N$-module and since $\mathcal{O}_N$ is $\mathcal{O}_M$-module, $\widetilde{\mathscr{F}}$ has the structure of an $\mathcal{O}_M$-module.

Let y be any element of $M-N$. Then there is an open neighborhood U of y in M such that $U \cap N = \emptyset$. Hence $\widetilde{\mathscr{F}}_U = 0$ and so $\widetilde{\mathscr{F}}|_{M-N}$ is wherent analytic sheaf. If $x \in N$ is an element, then x is an isolated point of $\phi^{-1}(\phi(x)) = \{x\}$. Then by Property 5, there are open neighborhood U of x in N and V of $\phi(x)$ in M such that $\phi(U) \subset V$, ϕ_{UV}: $U \longrightarrow V$ finite map and $\widetilde{\mathscr{F}}_V = (\phi_* \mathscr{F})_V$ coherent on V. Thus $\phi_* \mathscr{F} = \widetilde{\mathscr{F}}$ is coherent analytic sheaf on M.

($\Leftarrow$). We assume that $\phi_* \mathscr{F} = \widetilde{\mathscr{F}}$ is a coherent analytic sheaf on M. Then, for each $x \in N \subset M$, there exists an open neighborhood U of x in M such that

$$\mathcal{O}_U{}^p \longrightarrow \mathcal{O}_U{}^q \longrightarrow \widetilde{\mathscr{F}}_U \longrightarrow 0$$

is exact, where p and q are positive integers. Since ϕ^* is a right exact functor, we have the exact sequence of sheaves:

$$\phi^* \mathcal{O}_U{}^p \longrightarrow \phi^* \mathcal{O}_U \longrightarrow \phi^* \widetilde{\mathscr{F}}_U \longrightarrow 0$$

We put $N \cap \phi^{-1}(U) = V$. Then, since

$$\phi^* \mathcal{O}_U = \phi^{-1}\mathcal{O}_U \otimes_{\phi^{-1}\mathcal{O}_U} \mathcal{O}_N|_V = \mathcal{O}_N|_V \text{ and } \mathcal{O}_U{}^p = \mathcal{O}_U \oplus \cdots \oplus \mathcal{O}_U \ (p\text{-times}),$$

We have $\phi^*(\mathcal{O}_U{}^p) = (\mathcal{O}_N|_V)^p$

Thus

$$(\mathcal{O}_N|_V)^p \longrightarrow (\mathcal{O}_N|_V)^q \longrightarrow \phi^* \widetilde{\mathcal{F}}_U \longrightarrow 0$$

is exact. On the other hand,

$$\phi^* \widetilde{\mathcal{F}}_U = \phi^{-1}\mathcal{F}_U \otimes_{\phi^{-1}\mathcal{O}_U} \mathcal{O}_N|_V = \mathcal{F}_V \otimes_{\phi^{-1}\mathcal{O}_U} \mathcal{O}_N|_V$$

and

$$\begin{array}{ccc} \mathcal{F}_V \otimes_{\phi^{-1}\mathcal{O}_U} \mathcal{O}_N|_V & \xrightarrow{\ \sim\ } & \mathcal{F}_V \\ \Uparrow & & \Uparrow \\ f \otimes g & \longrightarrow & f \cdot g \end{array}$$

Therefore,

$$(\mathcal{O}_N|_V)^p \longrightarrow (\mathcal{O}_N|_V)^q \longrightarrow \mathcal{F}_V \longrightarrow 0$$

is exact ($\mathcal{F}$ is of finite representation). That is, $\mathcal{F}$ is a coherent analytic sheaf on N. ///

4. Supports of Coherent Sheaves

Let $\mathcal{F}$ be a sheaf on M. We define the support of $\mathcal{F}$, written Supp, ($\mathcal{F}$) by

$$\mathrm{Supp}(\mathcal{F}) = \{x \in M \mid \mathcal{F}_x \neq 0\}.$$

Proposition 4.1. If $\mathcal{F}$ is an analytic sheaf of finite type on M, then Supp ($\mathcal{F}$) is a closed subset of M.

Proof. Let $x \in M$–Supp($\mathcal{F}$) be any element. Then, since $\mathcal{F}$ is of finite type on M, there is an open neighnorhood U of x in M and $f_1, \cdots, f_k \in \mathcal{F}(U)$ such that $f_{1,y}, \cdots, f_{k,y}$ generates $\mathcal{F}_y$ for each $y \in U$. But since $x \notin \mathrm{Supp}(\mathcal{F})$, we have $\mathcal{F}_x = 0$ and $f_{1,x} \cdots = f_{k,x} = 0$. Hence there exist an open neighborhood V of x in U such that $f_1|_V = \cdots = f_k|_V = 0$. That is, $\mathcal{F}|_V = 0$ and so $V \subset M$–Supp($\mathcal{F}$). Thus M–Supp($\mathcal{F}$) is an open set in M, and so Supp($\mathcal{F}$) is a closed subset of M. ///

Theorem 4.2. Let $\mathcal{F}$ be a coherent analytic sheaf on M. Then Supp $\mathcal{F}$ is an analytic subset of M.

Proof. By our assumption, for an open subset U we have an exact sequence of sheaves:

$$\mathcal{O}_U{}^p \xrightarrow{s_0} \mathcal{O}_U{}^q \xrightarrow{s_1} \mathcal{F}_U \longrightarrow 0,$$

where p and q are positive integers. In this case, there exist $f_1, \cdots, f_q \in \mathcal{F}(U)$ such that, for each $x \in U$, $\{f_{1,x}, \cdots, f_{q,x}\}$ generates $\mathcal{F}_x$ and for $(a_{1,x}, \cdots, a_{q,x}) \in \mathcal{O}_x{}^q$

$$s_1(a_{1,x}, \cdots, a_{q,x}) = \sum_{i=1}^{q} a_{i,x} f_{i,x}.$$

Similarly, there exist $g_1, \cdots, g_p \in \mathcal{O}^q(U)$ such that $\{g_{1,x}, \cdots, g_{p,x}\}$ generates $\mathcal{R}el(f_1, \cdots, f_q)_x$ and for each $(b_{1,x}, \cdots, b_{p,x}) \in \mathcal{O}_x{}^p$

$$s_0(b_{1,x}, \cdots, b_{p,x}) = \sum_{j=1}^{p} b_{j,x} g_{j,x}.$$

We put

$$g_j = (g_j{}^1, \cdots, g_j{}^q) \in \mathcal{O}(U)^q, \quad (g_j{}^1, \cdots, g_j{}^q \in \mathcal{O}(U))$$

then s_0 can be written by a $q \times p$-matrix

$$s_0 = \begin{pmatrix} g_1{}^1, \cdots\cdots, g_p{}^1 \\ \vdots \qquad\qquad \vdots \\ g_1{}^q, \cdots\cdots, g_p{}^q \end{pmatrix}$$

That is, for each $x \in U$ and $(b_{1,x}, \cdots, b_{p,x}) \in \mathcal{O}_x{}^p$

$$s_{0,x}(b_{1,x}, \cdots b_{p,x}) = \begin{pmatrix} g^1_{1,x}, \cdots, g^1_{p,x} \\ \vdots \\ g^q_{1,x}, \cdots, g^q_{p,x} \end{pmatrix} \begin{pmatrix} b_{1,x} \\ \vdots \\ b_{p,x} \end{pmatrix}$$

$$= \begin{pmatrix} b_{1,x} g^1_{1,x} +, \cdots, + b_{p,x} g^1_{p,x} \\ \vdots \\ b_{1,x} g^p_{1,x} +, \cdots, + b_{p,x} g^q_{p,x} \end{pmatrix}$$

It is clear that, for each $i = 1, 2, \cdots, q$,

$$\{g^i_{1,x}, \cdots, g^i_{p,x}\} \text{ generates an ideal } \mathcal{Q}_{i,x} \text{ of } \mathcal{O}_x.$$

Noting that from the exact sequence

$$\mathcal{O}_U{}^p \xrightarrow{s_0} \mathcal{O}_U{}^q \xrightarrow{s_1} \mathcal{F}_U \longrightarrow 0,$$

we have, for each $x \in U$, the exact sequence

$$\mathcal{O}_x{}^p \xrightarrow{s_{0,x}} \mathcal{O}_x{}^q \xrightarrow{s_{1,x}} \mathcal{F}_x \longrightarrow 0,$$

and

$$\{x \in \mathrm{Supp}(\mathscr{F}) \cap U\} = \{x \in U \mid s_{1,x} \neq 0\}$$
$$= \{x \in U \mid \mathcal{O}_x/\mathscr{Q}_{1,x} \neq 0\} \cup \cdots \cup \{x \in U \mathcal{O}_x/\mathscr{Q}_{q,x} \neq 0\}$$

In facf,

$$\mathscr{F}_x = 0 \rightleftarrows s_{1,x} = 0 \rightleftarrows s_{0,x} \text{ is surjective} \rightleftarrows \mathscr{Q}_{1,x} = \cdots = \mathscr{Q}_{q,x} = \mathcal{O}_x$$

Recall that $\mathscr{M}_{\mathcal{O}_x}$ is the maximal ideal of $\mathcal{O}_x$(see §2). Then since

$$\mathcal{O}_x/\mathscr{Q}_{j,x} \neq 0 \rightleftarrows \mathscr{Q}_{j,x} \subset \mathscr{M}_{\mathcal{O}_x},$$

we have the following:

$$\mathcal{O}_x/\mathscr{Q}_{j,x} \neq 0 \rightleftarrows g_j^{1,x}, \cdots, g_j^{p,x} \text{ are not invertible } g_1^j(0) = \cdots = g_p^j(x) = 0.$$

Hence we have

$$\{x \in U \mid \mathcal{O}_x/\mathscr{Q}_{j,x} \neq 0\} = (g_1^j)^{-1}(0) \cap \cdots \cap (g_p^j)^{-1}(0)$$

and it is an analytic subset of M for $1 \leqq j \leqq q$. But since a finite union of analytic subsets is also an analytic subset, Supp($\mathscr{F}$) is an analytic subset of M./

Corollary 4.3. Let $\mathscr{F} \xrightarrow{a} \mathscr{G} \xrightarrow{b} \mathscr{K}$ be a sequence of coherent analytic sheaves on M such that

(i) $b \circ a = 0$

(ii) for a fixed point $x \in M$, $\mathscr{F} \xrightarrow{a_x} \mathscr{G} \xrightarrow{b_x} \mathscr{K}$ is exact.

Then the above sequence is exact on some open neighborhood of x.

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