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A Proof of Grace's Theorem by Induction

by

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1. Introduction.

At the the turn of the century J.H. Grace [1] introduced the following

Definition 1. Two polynomials

$$A(z)=a_0+\binom{n}{1}a_1z+\ldots+\binom{n}{k}a_kz^k+\ldots+a_nz^n \tag{1.1}$$

and

$$B(z)=b_0+\binom{n}{1}b_1z+\ldots+\binom{n}{k}b_kz^k+\ldots+b_nz^n \tag{1.2}$$

are said to be apolar provided that their coefficients satisfy the apolarity condition

$$a_0b_n-\binom{n}{1}a_1b_{n-1}+\ldots+(-1)^k\binom{n}{k}a_kb_{n-k}+\ldots+(-1)^na_nb_0=0. \tag{1.3}$$

The coefficients of the polynomials may be real or complex. If $a_r\neq 0$ $(r\geqq 0)$ and $a_\nu=0$ for $\nu=r+1, r+2,\ldots,n$, then we regard $z=\infty$ as an $(n-r)$-fold zero of $A(z)$. If all the coefficients of $A(z)$ are zero, then $A(z)$ is not regarded as a polytomial.

Grace discovered the following remarkable

Theorem of Grace. Let the polynomials (1.1) and (1.2) be apolar. Let $\alpha_1,\alpha_2,\ldots,\alpha_n$ be the zeros of $A(z)$ and $\beta_1,\beta_2,\ldots,\beta_n$ be the zeros of $B(z)$. If the circular region C oontains all of the α_ν, then C must contain at least one of the β_ν.

By a circular region we mean either the closed interior of a circle, or the closed exterior of a circle, or a closed half-plane.

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In [3] G. Szegö gave a proof of Grace's theorem freed of the invariant-theoretic concepts used by Grace in [1], and he also gave a large number of applications. In the present note we establish Grace's theorem by induction on n. Our proof is different from those given earlier.

2. The Invariance of Apolarity by Möbius Transformations.

By the transform of $A(z)$ under the Möbius transformation

$$(2.1)\qquad z=\frac{aw+b}{cw+d},\quad (ad-bc\neq 0),$$

we mean the polynomial

$$A^*(w)\equiv(cw+d)^n A\left(\frac{aw+b}{cw+d}\right)\equiv\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu(aw+b)^\nu(cw+d)^{n-\nu}$$

$$\equiv\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu{}^*w^\nu.$$

For example if $A(z)\equiv 1$, then $A^*(w)=(cw+d)^n$ and the n-fold zero of $A(z)$ at $z=\infty$ becomes an n-fold zero of $A^*(z)$ at $w=-d/c$ if $c\neq 0$.

Lemma 1. Let $A(z)$ and $B(z)$ be apolar polynomials. If the Möbius transformation (2.1) changes the polynomials (1.1) and (1.2) into

$$(2.2)\qquad A^*(w)=\sum_{\nu=0}^{n}\binom{n}{\nu}a_\nu{}^*w^\nu \text{ and } B^*(w)=\sum_{\nu=0}^{n}\binom{n}{\nu}b_\nu{}^*w^\nu,$$

then the polynomials (2.2) are also apolar.

Proof. It suffices to prove Lemma 1 for each of the three special transformations

$$(2.3)\qquad \text{(i) } z=w+h,\ \text{(ii) } z=kw,\ \text{(iii) } z=\frac{1}{w}.$$

$$\text{(i) } A^*(w)=A(w+h)=\sum_{\nu=0}^{n}\frac{w^\nu}{\nu!}A^{(\nu)}(\mathrm{h})$$

and therefore

$$A^*(w)=\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(n-\nu)!}{n!}A^{(\nu)}(h)w^\nu.$$

Similarly

$$B^*(w)=\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(n-\nu)!}{n!}B^{(\nu)}(h)w^\nu.$$

The apolarity equation for these polynomials is

$$f(h)=\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\frac{(n-\nu)!}{n!}A^{(\nu)}(h)\frac{\nu!}{n!}B^{(n-\nu)}(h)=0$$

or

$$(2.4)\qquad n!\ f(h)=\sum_{\nu=0}^{n}(-1)^{\nu}A^{(\nu)}(h)B^{(n-\nu)}(h)=0.$$

The apolarity of $A(z)$ and $B(z)$ gives $f(0)=0$, and we must show that $f(h)=0$ for all h. This will follow as soon as we show that for all h

$$(2.5)\qquad f'(h)=0.$$

From (2.4) we find that

$$n!\ f'(h)=\sum_{\nu=0}^{n}(-1)^{\nu}A^{(\nu+1)}(h)B^{(n-\nu)}(h)$$
$$+\sum_{\nu=0}^{n}(-1)^{\nu}A^{(\nu)}(h)B^{(n-\nu+1)}(h).$$

Here the ν th term $(\nu<n)$ in the first sum cancels with the $(\nu+1)$-st term in the second sum, and hence

$$n!\ f'(h)=(-1)^{n}A^{(n+1)}(h)B(h)+A(h)B^{(n+1)}(h),$$

which is evidently zero because $A(z)$ and $B(z)$ are nth degree polynomials. This proves (2.5) and therefore (2.4) for all h.

(ii) For the second transformation in (2.3) we have

$$A^{*}(w)=a_0+\binom{n}{1}a_1kw+\ldots+a_nk^nw^n,$$

and

$$B^{*}(w)=b_0+\binom{n}{1}b_1kw+\ldots+b_nk^nw^n$$

which are evidently apolar by (1.3).

(iii) Finally, setting $z=1/w$ gives

$$A^{*}(w)=a_n+\binom{n}{1}a_{n-1}w+\ldots+a_0w^n$$

and

$$B^{*}(w)=b_n+\binom{n}{1}b_{n-1}w+\ldots+b_0w^n$$

and these are also apolar by (1.3). ///

Lemma 2. If α is a zero of the polynomial $A(z)$, then its transform β under (2.1) is a zero of the transformed polynomial $A^*(w)$.

If neither α nor β is ∞, then $\alpha=(a\beta+b)/(c\beta+d)$ and

$$(2.6)\qquad A^*(\beta)=(c\beta+d)^n A\left(\frac{a\beta+b}{c\beta+d}\right)=(c\beta+d)^n A(\alpha)=0.$$

If $\alpha=\infty$ is an r-fold zero of $A(z)$, then $\beta=-d/c$ is clearly an r-fold zero of $A^*(z)$. If $\alpha=a/c$ is an r-fold zero of $A(z)$, then the decomposition used in the proof of Lemma 1 shows that $\beta=\infty$ is an r-fold zero of $A^*(z)$. ///

It follows from Lemma 2 that if a circular domain C contains all the zeros of $A(z)$ then the transformed domain under (2.1) will contain all the zeros of $A^*(z)$.

3. Proof of Grace's Theorem.

We use induction on n. For $n=1$, the apolarity condition (1.3) gives $a_0b_1-a_1b_0=0$ so $\alpha_1=\beta_1$ and the theorem is obviously true.

Next we assume the theorem is true for index n-1 and wish to prove that it is also true for index n. Here we use the method of contradiction. We shall assume that for some circular domain C and some pair of apolar polynomials $A(z)$ and $B(z)$

$$(3.1)\qquad \alpha_\nu\in C, \nu=1,2,\ldots,n, \text{ and } \beta_\nu\notin C, \nu=1,2,\ldots,n.$$

By a transformation we may assume that $\beta_n=\infty$, without loss of generality (use Lemmas 1 and 2). It follows that in (1.2)

$$(3.2)\qquad b_n=0.$$

The second assumption in (3.1) tells us that $\beta_n\notin C$ and hence C is bounded. Therefore all α_ν are finite and hence $a_n\neq 0$. The points $\beta_1,\beta_2,\ldots,\beta_{n-1}$ (finite or not) are the zeros of

$$(3.3)\qquad B(z)=b_0+\binom{n}{1}b_1z+\ldots+\binom{n}{k}b_kz^k+\ldots+\binom{n}{n-1}b_{n-1}z^{n-1}$$

which we now regard as a polynomial of degree n-1. Now consider the polynomial

$$(3.4)\qquad \frac{1}{n}A'(z)=a_1+\binom{n-1}{1}a_2z+\ldots+\binom{n-1}{k}a_{k+1}z^k+\ldots+a_nz^{n-1}$$

having the zeros $\gamma_1,\gamma_2,\ldots,\gamma_{n-1}$. These zeros are all finite because $a_n\neq 0$.

We claim the two polynomials (3.3) and (3.4) are apolar as polynomials of degree

$n-1$. To confirm this we rewrite (3.3) in the usual form

(3.5) $$B(z)=b_0'+\binom{n-1}{1}b_1'z+\dots+\binom{n-1}{k}b_k'z^k+\dots+b'_{n-1}z^{n-1}.$$

Then

(3.6) $$\binom{n}{k}b_k=\binom{n-1}{k}b_k',\quad k=0,1,2,\dots,n-1.$$

But then our original apolarity condition (1.3)

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}b_k a_{n-k}=0$$

(since $b_n=0$ by (3.2)) becomes

$$\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}b_k' a_{n-k}=0.$$

This shows that the polynomials (3.4) and (3.5) are apolar.

We now appeal to the Gauss-Lucas Theorem which states that all the zeros $\gamma_1,\gamma_2,\dots,\gamma_{n-1}$ are in the convex hull of the zeros $\alpha_1,\alpha_2,\dots,\alpha_n$ of $A(z)$. By our first assumption (3.1) we conclude that $\gamma_\nu\in C$, for $\nu=1,2,\dots,n-1$. On the other hand $\beta_\nu\notin C$ for $\nu=1,2,\dots,n-1$. This contradicts Grace's Theorem for index $n-1$. Hence by the principle of mathematical induction Grace's Theorem is true for every positive integer n. ///

4. Some Applications.

Grace's Theorem has many interesting applications. For brevity, we state some of these as Theorems. The reader may regard the proofs as exercises, or he (she) may find the proofs in Szegö's paper [3] or the book by Marden [2].

In the following theorems, $A(z)$ and $B(z)$ are defined by (1.1) and (1.2), and $C(z)$ is the related polynomial

(4.1) $$C(z)\equiv\sum_{k=0}^{n}\binom{n}{k}a_k b_k z^k.$$

We always assume that $A(z)$ and $B(z)$ are apolar.

Theorem A. If w is a zero of $C(z)$, and

(4.2) $$B^*(z)=z^nB\left(-\frac{w}{z}\right),$$

then $A(z)$ and $B^*(z)$ are apolar.

Theorem B. Call $P(z)=0$ an I equation if all its roots are in $|z|<1$, an $\bar{I}$ equation if all its roots are in $|z|\leqq 1$, an E equation if all its roots are in $|z|>1$, an $\bar{E}$ equation if all its roots are in $|z|\geqq 1$, and a U equation if all its roots are on $|z|=1$. If $A(z)=0$ is an I, E, or U equation and $B(z)=0$ is an $\bar{I}$, $\bar{E}$, or $\bar{U}$ equation (respectively), then $C(z)=0$, defined by (4.2) is an I, E, or U equation respectively.

Theorem C. If all the zeros of $A(z)$ are in $|z|<r$ and all the zeros of $B(z)$ are in $|z|\leqq\rho$, then all the zeros of $C(z)$ are in $|z|<r\rho$.

Theorem D. If all the zeros of $A(z)$ are in a closed and bounded convex domain D and all the zeros of $B(z)$ are in $[-1,0]$, then all the zeros of $C(z)$ lie in D.

Theorem E. An extension of Rolle's Theorem to complex functions. Let $P(z)$ be a polynomial of degree n and suppose that $P(1)=0$ and $P(-1)=0$. Then the derivative $P'(z)$ has a zero in $|z|\leqq\cot(\pi/n)$ and this result is sharp (best possible).

Further $P'(z)$ has a zero in Re $z\geqq 0$ and a zero in Re $z\leqq 0$.

References

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