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TOPOLOGICAL INVERSE SEMIGROUPS OF HOMOMORPHISMS

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1. Introduction

A topological inverse semigroup is a topological semigroup S which is (algebraically) an inverse semigroup and the inversion function on S is continuous. This concept was introduced by Eberhart, C. and Selden, J. [3]. And they established several properties of such objects, and investigated the structure of the closure of bicycle semigroup considered as a subsemigroup of a locally compact topological inverse semigroup. Selden, A.A. [7] investigated the structure of the closure of bisimple ω-semigroup considered as a subsemigroup of a locally compact topological inverse semigroup. In recent years, Koray, S. [5] gave a characterization of a certain subclass of bisimple locally compact topological inverse semigroup whose maximal subgroups are compact and whose set of idempotents is isomorphic to an interval of the real line.

In this papers, we investigate another properties of a topological inverse semigroup. And we discuss topological inverse semigroups of continuous functions and continuous homomorphisms relative to the compact-open topology and the pointwise multiplication.

2. Topological inverse semigroups

We begin with some definitions and notations for semigroups. A semigroup is a nonempty set S together with an associative multiplication $(x,y)\to xy$ from $S\times S$ into S. A semigroup S is said to be abelian if $ab=ba$ for all $a,b\in S$. A subsemigroup of a semigroup S is a nonempty subset T of S such that $T^2\subset T$. An element e of a semigroup S is called an idempotent if $e^2=e$. $E(S)$ will denote the set of all idempotents of S. A function $h: S\to T$ is called a homomorphism if $h(xy)=h(x)h(y)$ for all $x,y\in S$.

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DEFINITION 2.1. A topological semigroup is a Hausdorff space S together with a continuous associative multiplication.

The condition that multiplication on S is continuous is equivalent to the condition that for each $x, y \in S$ and each open set W in S with $xy \in W$, there exist open sets U and V in S such that $x \in U$, $y \in V$ and $UV \subset W$.

Throughout, all topological spaces will be assumed Hausdorff. If A is a subset of a space S, then $\bar{A}$ will denote the closure of A in S.

DEFINITION 2.2. A semigroup S is (algebraically) an inverse semigroup if each element x of S has a unique inverse; that is, there is a unique element x^{-1} of such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$.

A subsemigroup H of an inverse semigroup S is called an inverse subsemigroup if $x \in H$ implies $x^{-1} \in H$.

Note that if S is an inverse semigroup then xx^{-1} and $x^{-1}x$ are idempotents of S for each $x \in S$, and $(ab)^{-1}=b^{-1}a^{-1}$ for all $a, b \in S$ ([4]).

DEFINITION 2.3. A topological inverse semigroup is a topological semigroup S which is an inverse semigroup and the inversion function $x \rightarrow x^{-1}$ on S is continuous.

EXAMPLES([3]). Any topological group or any topological semilattice is a topological invese semigroup. On the other hand, the nonnegative real numbers with the usual topology and ordinary multiplication is an inverse semigroup which is not a topological inverse semigroup. Because of this, it is not true that if S is a locally compact topological semigroup which is an inverse semigroup then S is a topological inverse semigroup.

LEMMA 2.4([3]). *Let A be an inverse subsemigroup of a topological inverse semigroup S. Then A and $\bar{A}$ are topological inverse semigroups.*

DEFINITION 2.5. A topological group is a topological semigroup G which is (algebraically) a group and the inversion function $x \rightarrow x^{-1}$ on G is continuous.

THEOREM 2.6. *Let S be a topological inverse semigroup and let G be a subgroup of S. Then $\bar{G}$ is a topological subgroup of S.*

Proof. Since G is an inverse subsemigroup of S, by Lemma 2.4, $\bar{G}$ is a topological inverse subsemigroup of S. Let e be the identity of G.

Let $x \in \bar{G}$. Then there is a net $\{x_\alpha\}$ in G such that $\{x_\alpha\} \to x$. Using continuity of multiplcation on G, $\{x_\alpha\} = \{ex_\alpha\} \to ex$ and $\{x_\alpha\} = \{x_\alpha e\} \to xe$. Hence $ex = x = xe$. This implies that e is the identity of $\bar{G}$. Since the inversion function on G is continuous, $\{x_\alpha^{-1}\} \to x^{-1} \in \bar{G}$. So $\{e\} = \{x_\alpha x_\alpha^{-1}\} \to xx^{-1} \in \bar{G}$ and $\{e\} = \{x_\alpha^{-1} x_\alpha\} \to x^{-1}x \in \bar{G}$, and so $xx^{-1} = e = x^{-1}x$ for $x \in \bar{G}$. It follows that $\bar{G}$ is a topological group.

DEFINITION 2.7. Let $\{S_i\}_{i\in I}$ be a collection of topological semigroups. The coordinatewise multiplication on the product $\Pi\{S_i\}_{i\in I}$ is given by $(fg)(j) = f(j)g(j)$ for all $f, g \in \Pi\{S_i\}_{i\in I}$ and each $j \in I$, the latter product being taken is S_j for each $j \in I$.

THEOREM 2.8([3]). *Let $\{S_i\}_{i\in I}$ be a collection of topological semigroups. Then $\Pi\{S_i\}_{i\in I}$ with coordinatewise multiplication is a topological semigroup relative to the product topology.*

THEOREM 2.9. *Let $\{S_i\}_{i\in I}$ be a collection of topological inverse semigroups. Then $\Pi\{S_i\}_{i\in I}$ with coordinatewise multiplication is a topological inverse semigroup relative to the product topology.*

Proof. In view of Theorem 2.8., $\Pi\{S_i\}_{i\in I}$ is a topological semigroup. We establish that $\Pi\{S_i\}_{i\in I}$ is an inverse semigroup and the inversion function on $\Pi\{S_i\}_{i\in I}$ is continuous. Let $\phi_j : S_j \to S_j$ be the inversion function for each $j \in I$. For each $f \in \Pi\{S_i\}_{i\in I}$, define $\bar{f} = \phi_j \circ f$ for each $j \in I$, that is $\bar{f}(j) = f(j)^{-1}$ for each $j \in I$. Then $\bar{f} \in \Pi\{S_i\}_{i\in I}$ and $(f\bar{f}f)(j) = f(j)\bar{f}(j)f(j) = f(j)f(j)^{-1}f(j) = f(j)$ and $(\bar{f}f\bar{f})(j) = \bar{f}(j)f(j)\bar{f}(j) = f(j)^{-1}f(j)f(j)^{-1} = f(j)^{-1} = \bar{f}(j)$ for each $j \in I$. Thus $f\bar{f}f = f$ and $\bar{f}f\bar{f} = \bar{f}$. Hence $\bar{f}$ is an inverse of $f \in \Pi\{S_i\}_{i\in I}$. To prove the uniqueness of $\bar{f}$, we assume that $f\bar{\bar{f}}f = f$ and $\bar{\bar{f}}f\bar{\bar{f}} = \bar{\bar{f}}$ for some $\bar{\bar{f}} \in \Pi\{S_i\}_{i\in I}$. Then $f(j) = (f\bar{\bar{f}}f)(j) = f(j)\bar{\bar{f}}(j)f(j)$ and $\bar{\bar{f}}(j) = (\bar{\bar{f}}f\bar{\bar{f}})(j) = \bar{\bar{f}}(j)f(j)\bar{\bar{f}}(j)$ for each $j \in I$. Thus $\bar{\bar{f}}(j)$ is an inverse of $f(j)$. By uniqueness of $f(j)^{-1}$, $\bar{\bar{f}}(j) = f(j)^{-1} = \bar{f}(j)$ for each $j \in I$. Hence $\bar{\bar{f}} = \bar{f}$. Therefore, $\Pi\{S_i\}_{i\in I}$ is an inverse semigroup. Now, let $\psi : \Pi\{S_i\}_{i\in I} \to \Pi\{S_i\}_{i\in I}$ given by $\psi(f) = \bar{f}$ be the inversion function on $\Pi\{S_i\}_{i\in I}$. Consider the following diagram,

$$\begin{array}{ccc} \Pi\{S_i\}_{i\in I} & \xrightarrow{\ \psi\ } & \Pi\{S_i\}_{i\in I} \\ \Pi_j \downarrow & & \uparrow \Pi_j \\ S_j & \xrightarrow[\ \phi_j\ ]{} & S_j \end{array}$$

where Π_j denote the jth projection for each $j\in I$. For each $f\in\Pi\{S_i\}_{i\in I}$ and each $j\in I$, $(\Pi_j\circ\psi)(f)=\Pi_j(\psi(f))=\Pi_j(\bar{f})=\bar{f}(j)=f(j)^{-1}$ and $(\phi_j\circ\Pi_j)(f)=\phi_j(\Pi_j(f))=\phi_j(f(j))=f(j)^{-1}$. Hence the diagram commutes. Since $\phi_j\circ\Pi_j$ is continuous for each $j\in I$, $\Pi_j\circ\psi$ is continuous. Hence ψ is continuous. Therefore, $\Pi\{S_i\}_{i\in I}$ is a topological inverse semigroup.

3. Topological inverse semigroups of homomorphisms

We begin with some topological preliminaries. If X and Y are Hausdorff spaces, then $C(X, Y)$ denote the set of all continuous functions from X into Y. For $A\subset X$ and $B\subset Y$, we use $N(A, B)$ to denote the set $\{f\in C(X, Y)\mid f(A)\subset B\}$. The topology on $C(X, Y)$ having the collection of all $N(K, U)$ such that K is a compact subset of X and U is an open subset of Y as a subbase is called the compact-open topology on $C(X, Y)$.

LEMMA 3.1([2]). *Let X and Y be Hausdorff spaces. Then $C(X, Y)$ with the compact-open topology is a Hausdorff space.*

DEFINITION 3.2. Let S and T be topological semigroups. The pointwise multiplication on $C(S, T)$ is defined by $(fg)(x)=f(x)g(x)$ for all $f, g\in C(S, T)$ and all $x\in S$.

THEOREM 3.3. *Let S be a locally compact topological semigroup and let T be a topological semigroup. Then $C(S, T)$ with the pointwise multiplication is a topological semigroup relative to the compact-open topology.*

Proof. In view of Lemma 3.1, $C(S, T)$ is a Hausdorff space. Let $f, g, h\in C(S, T)$. Then we easily have $(fg)h=f(gh)$. Hence $C(S, T)$ is a semigroup. To prove the multiplication on $C(S, T)$ is continuous, let K be a compact subset of S, W an open subset of T. Suppose $f, g\in C(S, T)$ such that $fg\in N(K, W)$. Then $(fg)(K)\subset W$. Hence $(fg)(x)=f(x)g(x)\in W$ for each $x\in K$. By continuity of multiplication on T, for each $x\in K$, there exist open sets U_x and V_x in T such that $f(x)\in U_x$, $g(x)\in V_x$ and $U_xV_x\subset W$. Since S is locally compact and f and g are continuous, for each $x\in K$, there exist open sets A_x and C_x a compact set in S such that $x\in A_x\subset\bar{A}_x\subset C_x$, $f(C_x)\subset U_x$ and $g(C_x)\subset V_x$. Then $\{A_x\mid x\in K\}$ is an open cover of K. Since K is compact, there exists a finite cover $A_1, \cdots, A_n$ of A_x's and corresponding $\bar{A}_1, \cdots, \bar{A}_n$ of $\bar{A}_x$'s. Then $\bar{A}_1, \cdots, \bar{A}_n$ is a cover of K by compact sets. Let $U_1, \cdots, U_n$ and $V_1, \cdots, V_n$

be the corresponding collections of U_x's and V_x's respectively. Then for each $1 \leqq j \leqq n$, we have $f(\bar{A}_j) \subset U_j$, $g(\bar{A}_j) \subset V_j$ and $U_j V_j \subset W$. Hence $f \in \bigcap_{j=1}^{n} N(\bar{A}_j, U_j)$ and $g \in \bigcap_{j=1}^{n} N(\bar{A}_j, V_j)$. Suppose that $h \in \bigcap_{j=1}^{n} (\bar{A}_j, U_j)$ and $k \in \bigcap_{j=1}^{n} N(\bar{A}_j, V_j)$ and fix $x \in K$. Then $x \in \bar{A}_i$ for some $1 \leqq i \leqq n$. So $(hk)(x) = h(x)k(x) \in h(\bar{A}_i)k(\bar{A}_i) \subset U_i V_i \subset W$. It follows that $hk \in N(K, W)$. Hence $(\bigcap_{j=1}^{n} N(\bar{A}_j, U_j))(\bigcap_{j=1}^{n} N(\bar{A}_j, V_j)) \subset N(K, W)$, and hence the multiplication on $C(S, T)$ is continuous. Therefore $C(S, T)$ is a topological semigroup.

COROLLARY 3.4. *Let S be a locally compact topological semigroup and let T be an abelian topological semigroup. Then $C(S, T)$ is an abelian topological semigroup.*

Proof. Let $f, g \in C(S, T)$ and let $x \in S$. Then $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$. Hence $C(S, T)$ is abelian.

THEOREM 3.5. *Let S be a locally compact topological semigroup and let T be a topological inverse semigroup. Then $C(S, T)$ with pointwise multiplication is a topological inverse semigroup.*

Proof. In view of Theorem 3.3, $C(S, T)$ is a topological semigroup. We establish that $C(S, T)$ is an inverse semigroup and the inversion function on $C(S, T)$ is continuous. Let $\phi : T \to T$ be the inversion function. For each $f \in C(S, T)$, let $\bar{f} = \phi \circ f$, that is, $\bar{f}(x) = f(x)^{-1}$ for all $x \in S$. Then $\bar{f} \in C(S, T)$. For $x \in S$, $(f\bar{f}f)(x) = f(x)\bar{f}(x)f(x) = f(x)f(x)^{-1}f(x) = f(x)$ and $(\bar{f}f\bar{f})(x) = \bar{f}(x)f(x)\bar{f}(x) = f(x)^{-1}f(x)f(x)^{-1} = f(x)^{-1} = \bar{f}(x)$. Hence $f\bar{f}f = f$ and $\bar{f}f\bar{f} = \bar{f}$. We assume that $f\bar{\bar{f}}f = f$ and $\bar{\bar{f}}f\bar{\bar{f}} = \bar{\bar{f}}$ for some $\bar{\bar{f}} \in C(S, T)$. Then $f(x) = (f\bar{\bar{f}}f)(x) = f(x)\bar{\bar{f}}(x)f(x)$ and $\bar{\bar{f}}(x) = (\bar{\bar{f}}f\bar{\bar{f}})(x) = \bar{\bar{f}}(x)f(x)\bar{\bar{f}}(x)$ for all $x \in S$. Thus $\bar{\bar{f}}(x)$ is an inverse of $f(x)$. By uniqueness of the inverse of $f(x)$, $\bar{\bar{f}}(x) = f(x)^{-1} = \bar{f}(x)$ for all $x \in S$. Hence $\bar{\bar{f}} = \bar{f}$, and hence $C(S, T)$ is an inverse semigroup. Now, let $\rho : C(S, T) \to C(S, T)$ be the inversion function. Then $\rho(f) = \bar{f} = \phi \circ f$. Let K be a compact subset of S, W an open subset of T, $f \in C(S, T)$ and $\bar{f} = \rho(f) \in N(K, W)$. Then $(\phi \circ f)(K) = \bar{f}(K) = \rho(f)(K) \subset W$. Hence $f(K) \subset \phi^{-1}(W)$, and hence $f \in N(K, \phi^{-1}(W))$, where $\phi^{-1}(W)$

is an open subset of T because of the inversion function ϕ is continuous. If $g\in N(K,\phi^{-1}(W))$, then $g(K)\subset\phi^{-1}(W)$. So $\rho(g)(K)=\bar{g}(K)=(\phi\circ g)(K)\subset W$, and so $\rho(g)\in N(K, W)$. Thus $\rho(N(K, \phi^{-1}(W))\subset N(K, W)$ Hence ρ is continuous. Therefore, $C(S, T)$ is a topological inverse semigroup.

COROLLARY 3.6. *Let S be a locally compact topological semigroup and let T be an abelian topological inverse semigroup. Then $C(S, T)$ is an abelian topological inverse semigroup.*

Let S be a semigroup and let T be an abelian semigroup. If f and g are homomorphisms from S into T and fg is defined by $(fg)(x)=f(x)g(x)$ for all $x\in S$ (pointwise multiplication), then fg is also a homomorphism from S into T. Hence if the set of homomorphisms from S into T is nonempty then it is an abelian semigroup under the pointwise multiplication.

Let S be a topological semigroup and let T be an abelian topological semigroup. Hom(S, T) denotes the set of all continuous homomorphisms from S into T with pointwise multiplication and the compact-open topology.

REMARK. Let S be a topological semigroup and let T be an abelian topological invese semigroup. Then Hom$(S, T)\neq 0$. For, $e\in E(T)$ and $f: S\rightarrow T$ the constant map with image $\{e\}$ imply $f\in$Hom(S, T). In particular, if T is a compact semigroup or T is a group then Hom$(S, T)\neq 0$. Thus we have following corollaries.

COROLLARY 3.7. *Let S be a locally compact topological semigroup and let T be an abelian topological semigroup such that* Hom$(S, T)\neq 0$. *Then* Hom(S, T) *is an abelian topological semigroup.*

Proof. Hom(S, T) is a subsemigroup of the abelian topological semigroup $C(S, T)$. Hence Hom(S, T) is an abelian topological semigroup.

COROLLARY 3.8. *Let S be a locally compact topological semigroup and let T be an abelian topological inverse semigroup. Then* Hom(S, T) *is an abelian topological inverse semigroup.*

Proof. In view of Corollary 3.7, Hom(S, T) is a subsemigroup of the abelian topological inverse semigroup $C(S, T)$. Let $\phi: T\rightarrow T$ be the inversion function. For each $f\in$Hom(S, T), let $\bar{f}=\phi\circ f$, that is, $\bar{f}(x)$

$=f(x)^{-1}$ for all $x\in S$. Then $\bar{f}\in C(S, T)$ and $\bar{f}$ is the inverse of f. Let $x, y\in S$. Then $\bar{f}(xy)=f(xy)^{-1}=(f(x)f(y))^{-1}=f(y)^{-1}f(x)^{-1}=f(x)^{-1}f(y)^{-1}=\bar{f}(x)\bar{f}(y)$. Hence $\bar{f}\in\mathrm{Hom}(S, T)$, and hence $\mathrm{Hom}(S, T)$ is an inverse subsemigroup of the abelian topological inverse semigroup $C(S, T)$. By Lemma 2.4, $\mathrm{Hom}(S, T)$ is a topological inverse semigroup. Hence $\mathrm{Hom}(S, T)$ is an abelian topological inverse semigroup.

The following result is an analog to a compact abelian topological group T.

THEOREM 3.9. *Let S be a discrete semigroup and let T be a compact abelian topological inverse semigroup. Then* $\mathrm{Hom}(S, T)$ *is a compact abelian topological inverse semigroup.*

Proof. Since S is a discrete semigroup, S is a locally compact topological semigroup. By Corollary 3.8, $\mathrm{Hom}(S, T)$ is an abelian topological inverse semigroup. Suppose that K is a compact subset of S and W is an open subset of T. Since S is discrete, K is finite. Let $K=\{k_1, \cdots, k_n\}$ for some $n\in N$, where N denotes the set of positive integers. Then $f\in N(K, W)$ if and only if $f(K)\subset W$ if and only if $f(k_i)\in W$ for each $k_i\in K$, $(i=1, \cdots, n)$ if and only if $f\in N(\{k_i\}, W)$ for each $k_i\in K$ $(i=1, \cdots, n)$ if and only if $f\in\bigcap_{i=1}^{n}N(\{k_i\}, W)$. Hence $N(K, W)=\bigcap_{i=1}^{n}N(\{k_i\}, W)$. It follows that $\{N(\{p\}, W)\,|\,p\in S$ and W is an open subset of $T\}$ is a subbase for the topology of $\mathrm{Hom}(S, T)$. Let $T_p=T$ for each $p\in S$ and consider the compact space $X=\Pi\{T_p\}_{p\in S}$. Define $F:\mathrm{Hom}(S, T)\to X$ by $F(\phi)(x)=\phi(x)$ for all $\phi\in\mathrm{Hom}(S, T)$ and $x\in S$. Then F is injective. Now, for the subbasic open set $N(\{q\}, W)$ of $\mathrm{Hom}(S, T)$, we have $F(N(\{q\}, W))=F(\{\phi\in\mathrm{Hom}(S, T)\,|\,\phi(q)\in W\})=\{F(\phi)\,|\,\phi\in\mathrm{Hom}(S, T)$ and $\phi(q)\in W\}=\{F(\phi)\,|\,\phi\in\mathrm{Hom}(S, T)$ and $\Pi_q(\phi)\in W\}=\{F(\phi)\,|\,\phi\in\mathrm{Hom}(S, T)$ and $\phi\in\Pi_q^{-1}(W)\}=F(\mathrm{Hom}(S, T))\cap\Pi_q^{-1}(W)$. It follows that F maps $\mathrm{Hom}(S, T)$ homeomorphically into X. Since S is discrete, $F(\mathrm{Hom}(S, T))$ is precisely the set of elements of X which are homomorphisms of S into T. Now, we show that $F(\mathrm{Hom}(S, T))$ is closed in X. Fix $g\in X-F(\mathrm{Hom}(S, T))$ Then g is not a homomorphism from S into T. Thus $g(xy)\neq g(x)g(y)$ for some $x, y\in S$. Hence there exist open sets W_1, W_2 and W_3 in T such that $g(xy)\in W_1$, $g(x)\in W_2$, $g(y)\in W_3$ and $W_1\cap W_2W_3=\phi$. It follows that $g\in\Pi_{xy}^{-1}(W_1)\cap\Pi_x^{-1}(W_2)\cap\Pi_y^{-1}(W_3)\subset X-F(\mathrm{Hom}(S, T))$. Hence $F(\mathrm{Hom}(S, T))$ is a closed subspace of X.

Since $\mathrm{Hom}(S, T)$ is homeomorphic to $F(\mathrm{Hom}(S, T))$, $\mathrm{Hom}(S, T)$ is a closed subspace of X. Hence $\mathrm{Hom}(S, T)$ is compact.

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