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ON THE ORIENTABILITY AND OBSTRUCTION CLASSES

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The purpose of this paper is to summarize a result of our seminar about characteristic classes which was held during the last winter vacation. That is, in Theorem 6, we prove that for an n-dimensional vector bundle ξ over CW complex B, if there exists a cross section of ξ over the 1-skelton of B then ξ is orientable. And, in Theorem 7, we prove that for any n-dimensional vector bundle ξ over a CW complex space

$$\omega_1(\xi)=0 \Leftrightarrow \xi \text{ is orientable.}$$

Througout this paper, by a vector bundle we mean a real vector bundle. Let $\xi=(E(\xi), \pi_\xi, B(\xi))$ be an n-dimensional vector bundle. The expression $H^i(B(\xi);G)$ denotes the i-th singular cohomology group of $B(\xi)$ with coefficient group G.

For each vector bundle $\xi=(E(\xi), \pi_\xi, B(\xi))$, a sequence of cohomology classes $\omega_i(\xi)\in H^i(B(\xi);\boldsymbol{Z}/2)$, $i=0,1,2\cdots$, called the Stiefel-Whitney classes of ξ, is defined axiomatically by the follwoing conditions:

(i) $\omega_0(\xi)=1\in H^0(B(\xi);\boldsymbol{Z}/2)$ and $\omega^i(\xi)=0$ for $i>\dim(\xi)$.

(ii) If $f: B(\xi)\to B(\eta)$ is covered by a bundle map from ξ to η, then $\omega_i(\xi)=f^*(\omega_i(\eta))$ for $i=0,1,2,\cdots$.

(iii) The Whitney Product Theorem. If ξ and η are vector bundle over the same base space, then

$$\omega_k(\xi\oplus\eta)=\sum_{i+j=k}\omega_i(\xi)\cup\omega_j(\eta),$$

where $\cup$ means the cup product.

(iv) Let $P^1(R)$ be the one dimensional real projective space, and

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$\gamma_1^1=(L, \pi, P^1(R))$ be the Hopf line bundle $P^1(R)$. Then $\omega_1(\gamma_1^1)\neq 0$ ([1] and [2]).

Let $G_n=G_n(R^\infty)$ be the infinite Grassmann manifold, and let γ^n be the n-dimensional universal vector bundle over G_n. Then the cohomology ring $H^*(G_n;Z/2)=\sum_{i=0}^{\infty}H^i(G_n;Z/2)$ is a polynomial ring over $Z/2$ freely generated by $\omega_1(\gamma^n), \cdots, \omega_n(\gamma^n)$ ([1], [2]).

Let $\widetilde{G}_n=\widetilde{G}_n(R^\infty)$ denote the Grassmann manifold consisting of all oriented n-planes in R^∞. Let $\tilde{\gamma}^n$ be the n-dimensional oriented universal vector bundle over $\widetilde{G}_n$, then there exists the canonical bundle map

$$\begin{array}{ccc} E(\tilde{\gamma}^n) & \xrightarrow{\ \tilde{p}\ } & E(\gamma^n) \\ \downarrow & & \downarrow \\ \widetilde{G}_n & \xrightarrow[\ p\ ]{} & G_n, \end{array}$$

where $\gamma^n=(E(\gamma^n), \pi, G_n)$ and $\tilde{\gamma}^n=(E(\tilde{\gamma}^n), \tilde{\pi}, \widetilde{G}_n)$. Of course, $p:\widetilde{G}_n\to G_n$ is a 2-fold covering map. That is, if $+V$ is an n-dimensional vector space with usual orientation then we denote the n-dimensional vector space with the opposite orientation of $+V$ by $-V$. Then for each $V\in G_n$, $+V$ and $-V$ are elements of $\widetilde{G}_n$. In this case $p(\pm V)=V$.

PROPOSITION 1. *Let $\xi=(E(\xi), \pi_\xi, B(\xi))$ be an oriented n-dimensional vector bundle over a CW complex space $B(\xi)$. Then there exists a bundle map $f:\xi\to\gamma^n$ such that $f^*(\gamma^n)\cong\xi$ except orientation. Moreover, $f:\xi\to\gamma^n$ lifts uniquely to an orientation preserving bundle map $\tilde{f}:\xi\to\tilde{\gamma}^n$.*

Proof. Since $B(\xi)$ is paracompact there exists a locally finite covering of $B(\xi)$ by countably many open subsets $U_1, U_2, \cdots$, so that $\xi|U_i$ is trivial for each i ([2]). Since ξ is oriented there exists an orientation preserving map (for each fiber)

$$h_i:\pi_\xi^{-1}(U_i)\longrightarrow \boldsymbol{R}^n$$

which maps each fiber of $\xi|U_i$ linearly and onto $\boldsymbol{R}^n$. Sine $B(\xi)$ is normal there exists an open covering $V_1, V_2, \cdots$, of $B(\xi)$ such that $\overline{V}_i\subset U_i$ for each i, where $\overline{V}_i$ is the closure of V_i. For each i there exists an open subset W_i such that $\overline{W}_i\subset V_i$. Define a continuous map

$$\lambda_i : B(\xi) \longrightarrow R$$

such that $\lambda_i | \overline{W}_i = 1$, $\lambda_i | B - V_i = 0$ and $0 \leq \lambda_i \leq 1$ for all i

Define $h_i^1 : E(\xi) \to \boldsymbol{R}^n$ by

$$h_i^1(e) = \begin{cases} 0 & \text{if } \pi_\xi(e) \notin V_i \\ \lambda_i(\pi_\xi(e)) h_i(e) & \text{if } \pi_\xi(e) \in V_i \end{cases}$$

Then it is clear that i) h_i^1 is continuous, ii) h_i^1 linear on each fiber, and iii) h_i^1 is orientation preserving for each fiber. We also define

$$\hat{f} : E(\xi) \longrightarrow \boldsymbol{R}^n \oplus \boldsymbol{R}^n \oplus \cdots = \boldsymbol{R}^\infty$$

by $\hat{f}(e) = (h_1'(e), h_2'(e), \cdots)$. Then it is obvious that $\hat{f}$ is continuous and maps each fiber injectively. Since the covering $\{V_i : i = 1, 2, \cdots\}$ is locally finite, the components $h_i^1(e)$ of $\hat{f}(e)$ are zero except for a finite number of its components. Moreover, for each $e \in E(\xi)$

$$(\hat{f} \,|\, \text{the fiber through } e) \text{ is orientation preserving}$$

and $\hat{f}$(the fiber through e) is an n-dimensional vector space in $\boldsymbol{R}^\infty$ having an orientation. When we disregard the orientation of $\hat{f}$(the fiber through e), it is clear that $\hat{f}$(the fiber through e)$\in G_n$. Define $\bar{f} : E(\xi) \to E(\gamma^n)$ by $\bar{f}(e) = (f$(the fiber through e), $\hat{f}(e))$ and $f : B(\xi) \to G_n$ by $f(\pi_\xi(e)) = \hat{f}$ (the fiber through e), where in the image of f and $\bar{f}$ orientation is disregarded. Then $(\bar{f}, f) : \xi \to \gamma^n$ is a bundle map and $f^*(\gamma^n) = \xi$ except orientation. By the definition of $\hat{f}$ above, if we regard the orientation of $\hat{f}$(the fiber through e), then it is clear that

$$\hat{f}(\text{the fiber through } e) \in \widetilde{G}_n.$$

Therefore, if we define $\tilde{f} : B(\xi) \to \widetilde{G}_n$ by $\tilde{f}(\pi_\xi(e)) = \hat{f}$ (the fiber through e), then we have the commutaive diagram

$$\begin{array}{ccc} & & \widetilde{G}_n \\ & & \big\downarrow P \\ B(\xi) & \xrightarrow{\ f\ } & G_n \end{array}$$

In this case, the map $\bar{\bar{f}} : E(\xi) \to E(\tilde{\gamma}^n)$ defined by $\bar{\bar{f}}(e) = (\hat{f}$(the fiber through e), $\hat{f}(e))$ is well-defined, since $\hat{f}$ is orientation preserving.///

Recall the projection $P : \widetilde{G}_n \to G_n$. For each $V \in G_n$, $p^{-1}(V) = \pm V$. We want to construct a line bundle ξ over G_n as follows. The total space E

of ξ is obtained from $\tilde{G}_n \times R$ by identifying each pair $(+V, t)$, and $(-V, -t)$, where for each $V \in G_n$, $p^{-1}(V) = \pm V \in \tilde{G}_n$. Then there exists a canonical projection

$$\begin{aligned} \tilde{G}_n \times R &\longrightarrow E \\ (+V, t) &\to [+V, t] \ \ (= \{(+V, t),\ (-V, -t)\}). \end{aligned}$$

and the topology of E is the quotient topology of $\tilde{G}_n \times R$ by the above projection.

LEMMA 2. *In the above situation we have an exact sequence*

$$\cdots \to H^{j-1}(G_n; \mathbf{Z}/2) \xrightarrow{\cup \omega_1(\xi)} H^j(G_n; \mathbf{Z}/2) \xrightarrow{P^*} H^j(\tilde{G}_n; \mathbf{Z}/2) \longrightarrow$$
$$H^j(G_n; \mathbf{Z}/2) \xrightarrow{\cup \omega_1(\xi)} H^{j+1}(G_n; \mathbf{Z}/2) \longrightarrow H^{j+1}(\tilde{G}_n; \mathbf{Z}/2) \to \cdots$$

Proof. Put

$$E - G_n = \{[+V, t] \in E \mid t \neq 0\} = E_0,$$

and

$$[+V, 1] = +V, \ \ [+V, -1] = -V.$$

Then $\tilde{G}_n \subset E_0$. Furthermore, $\tilde{G}_n$ is a deformation retract of E_0, because there exists a homotopy $H: E_0 \times [0, 1] \to E_0$ defined by $H(([+V, t], s)) = \left[+V, \frac{(1-s)t}{|t|} + st\right]$. For each $V \in G_n$ the fiber of ξ at V is $[+V, \mathbf{R}] = \{[+V, t] \mid t \in R\}$. Thus the mapping $[+V, \mathbf{R}] \to \mathbf{R}$ via $[+V, t] \to t$ determines the orientation of ξ. Let $e(\xi)$ denote the Euler class of ξ. By using the equalities

$$e(\xi) = \omega_1(\xi),$$
$$H^*(E; \mathbf{Z}/2) = H^*(G_n; \mathbf{Z}/2)$$

and

$$H^*(E_0; \mathbf{Z}/2) = H^*(G_n; \mathbf{Z}/2)$$

in the Thom-Gysin sequence ([1]), we can get the desired long exact sequence in the Lemma.

PROPOSITION 3. $\omega_1(\tilde{\gamma}^n)=0$.

Proof. In the proof of Lemma 2, $\omega_1(\xi)\neq 0$. This can be proved as follows. From Lemma 2, we have the exact sequence

$$0\longrightarrow H^0(G_n;Z/2)\longrightarrow H^0(\tilde{G}_n;Z/2)\longrightarrow H^0(G_n;Z/2)\xrightarrow{\cup\omega_1(\xi)}\cdots$$

Since any n-dimensional vector space in R^∞ can be deformed continuously to any other oriented n-dimensional vector space we have

$$H^0(\tilde{G}_n;Z/2)\cong Z/2.$$

Moreover, since $H^0(G_n;Z/2)=Z/2$ we have the exact sequence

$$0\longrightarrow H^0(G_n;Z/2)\xrightarrow{\cup\omega_1(\xi)}H^1(G_n;Z/2)\ (\cong Z/2)\rightarrow\cdots$$

and thus $\omega_1(\xi)\neq 0$ (Note that $H^1(G_n;Z/2)=\{0,\ \omega_1(\gamma^n)\}$ as in the above descriptions). Therefore $\omega_1(\xi)=\omega_1(\gamma^n)$. From the above facts and Lemma 2, we have the exact sequence

$$0\longrightarrow H^1(\tilde{G}_n;Z/2)\rightarrow H^1(G_n;Z/2)\xrightarrow{\cup\omega_1(\gamma^n)}H^2(G_n;Z/2)\rightarrow\cdots$$

Note that $H^2(G_n;Z/2)=\{0,\ \omega_1(\gamma^n)\cup\omega_1(\gamma^n),\ \omega_2(\gamma^n)\}$. Thus

$$H^1(G_n;Z/2)\xrightarrow{\cup\omega_1(\gamma^n)}H^2(G_n;Z/2)$$

is a monomorphism, and hence $H^1(\tilde{G}_n;Z/2)=0$. Therefore

$$0=\omega_1(\tilde{\gamma}^n)\in H^1(\tilde{G}_n;Z/2)=0.\quad ///$$

In consequence, we can prove (cf. [2]) that

$$H^*(\tilde{G}_n;Z/2)=Z/2[\omega_2(\tilde{\gamma}^n),\cdots,\omega_n(\tilde{\gamma}^n)].$$

Let $\xi=(E,\pi,B)$ be an n-dimensional vector bundle, and let $\Lambda^n\xi$ be the one-dimensional exterior algebra bundle of ξ. Then, as is well konwn (cf. [3])

$$\Lambda^n\xi \text{ is trivial} \Leftrightarrow \xi \text{ is orientable.}$$

Moreover if B is a CW-complex, then (cf. [3])

$$\Lambda^n\xi \text{ is trivial} \Leftrightarrow \xi \text{ is orientable.}$$

And, if ξ is an one-dimensional vector bundle, then

$$\omega_1(\xi)=0 \Leftrightarrow \xi \text{ is trivial.}$$

LEMMA 4. *Let $\xi=(E, \pi, B)$ be an n-dimensional vector bundle. Then we have $\omega_1(\xi)=\omega_1(\Lambda^n\xi)$.*

Proof. From the splitting principle, there is a map $g: B_1 \to B$ ([3]) such that

(i) $g^*\xi$ is a Whitney sum of line bundles,
(ii) $g^*: H^*(B;\boldsymbol{Z}/2) \to H^*(B_1;\boldsymbol{Z}/2)$ is injective.

Let us put

$$g^*\xi=\xi_1^1 \oplus \cdots \oplus \xi_n^1$$

where each $\xi_k^1 (1 \leq k \leq n)$ is a line bundle over B_1. Note that for a 1-dimensional vector bundle

$$\omega_1(\xi)=e(\xi),$$

where $e(\xi)$ is the mod $\boldsymbol{Z}/2$ Euler class of ξ, and

$$e(\xi \otimes \xi')=e(\xi)+e(\xi'),$$

where ξ' is also a one-dimensional vector bundle having the same base space as ξ. Thus

$$\begin{aligned} g^*\omega_1(\xi) &= \omega_1(g^*\xi)=\omega_1(\xi_1^1 \oplus \cdots \oplus \xi_n^1) \quad \text{(Naturality)} \\ &= \omega_1(\xi_1^1)+\cdots+\omega_1(\xi_n^1) \quad \text{(Whitney product theorem)} \\ &= \omega_1(\xi_1^1 \otimes \cdots \otimes \xi_n^1) \\ &= \omega_1(\Lambda^n(g^*\xi)) \\ &= g^*\omega_1(\Lambda^n\xi). \end{aligned}$$

Therefore, we have $\omega_1(\xi)=\omega_1(\Lambda^n\xi)$. ///

Let $\xi=(E, \pi, B)$ be an n-dimensional vector bundle over a CW complex B. For each fiber F of ξ we put

$$V_n(F)=\{\text{all } n\text{-frames in } F\},$$

where an n-frame means an n-tuple $(v_1, \cdots, v_n)$ of linearly independent vectors of F. Then $V_n(F)$ is an open subset of $F \times \cdots \times F$ (n-times).

DEFINITION 5. In the above situation, the first primary obstruction class $O_1(\xi)$ of ξ is an element of $H^1(B;\widetilde{H}_0(V_n(F);\boldsymbol{Z}))$ such that

$$O_1(\xi)=0 \Leftrightarrow \text{there exists a cross section over the 1-skeleton of } B,$$

where $\widetilde{H}_0(V_n(F);\boldsymbol{Z})$ is the reduced singular homology group.

Since there exists a unique group homomorphism

$$h:\widetilde{H}_0(V_n(F);\boldsymbol{Z})\longrightarrow \boldsymbol{Z}/2,$$

we have $h_*(O_1(\xi))\in H^1(B;\boldsymbol{Z}/2)$. In this case,

$$\omega_1(\xi)=h_*(O_1(\xi))$$

([1], [2], and [3]).

THEOREM 6. *In the above situation ξ is orientable if there exists a cross section over the* 1*-skeleton of B.*

Proof. By our hypothesis $O_1(\xi)=0$, and thus $\omega_1(\xi)=0$. By Lemma 4, $\omega_1(\xi)=\omega_1(\Lambda^n\xi)=0$.

Since

$$\omega_1(\eta)=0 \rightarrow \eta \text{ is trivial}$$

if η is a line bundle ([2]), $\Lambda^n\xi$ is trivial. Therefore ξ is orientable.

THEOREM 7. *For an n-dimensional vector bundle $\xi=(E,\pi,B)$ over a CW complex space B,*

$$\omega_1(\xi)=0 \Leftrightarrow \xi \textit{ is orientable.}$$

Proof. In the proof of Theorem 6, we have already proved the part $(\Rightarrow)$. Since B is paracompact and ξ is orientable, we have a bundle map

$$\begin{array}{ccc} E & \xrightarrow{\ \bar{f}\ } & E(\tilde{\gamma}^n) \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ B & \xrightarrow{\ f\ } & \widetilde{G}_n. \end{array}$$

And, by Proposition 3, $\omega_1(\tilde{\gamma}^n)=0$, therefore we have

$$\omega_1(\xi)=f^*(\omega_1(\tilde{\gamma}^n))=0.$$

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