Comm. Korean Math. Soc. 2 (1987), No. 2, pp. 235~242

ON FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS ON METRIC SPACES

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1. Introduction

The Banach contraction principle has numerous generalizations. One of these is due to G.E. Hardy and T.D. Rogers in 1973. They proved the following result ([4]).

THEOREM A. *Let T be a single-valued self-mapping on a metric space (X, d), Suppose that there exist nonnegative constants a, b, c, e, f such that for $x, y \in X$*

(1) $$d(Tx, Ty) \leq a \cdot d(x, Tx) + b \cdot d(y, Ty) + c \cdot d(x, Ty) + e \cdot d(y, Tx) + f \cdot d(x, y).$$

Set $\alpha = a+b+c+e+f$. Then

(a) *if X is complete and $\alpha < 1$, then T has a unique fixed point;*

(b) *if (1) modified to the condition $x \neq y$ implies that*

(1') $$d(Tx, Ty) < a \cdot d(x, Tx) + b \cdot d(y, Ty) + c \cdot d(x, Ty) + e \cdot d(y, Tx) + f \cdot d(x, y)$$

and in this case if X is compact, T is continuous and $\alpha = 1$, then T has a unique fixed point.

Soon after, C.S. Wong generalized the above result as follows ([15]).

THEOREM B. *Let T be a single-valued self-mapping on a complete metric space (X, d). Suppose that there exist functions α_i, $i = 1, 2, \ldots, 5$, of $(0, \infty)$ into $[0, \infty)$ such that*

(a) *each α_i is upper semicontinuous from the right, (i.e., $\operatorname{Lim}\operatorname{Sup}_{n \to \infty} \alpha_i(b_n) \leq \alpha_i(b)$ for each decreasing convergent sequence $\{b_n\}$ with limit b),*

(b) $\sum_{i=1}^{5} \alpha_i(t) < t$, $t > 0$, *and*

Received August 18, 1987.

(c) *for any distinct* $x, y \in X$ *we have*

$$(2)\quad d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Tx) + a_5 d(x, y)$$

where $a_i = \alpha_i(d(x, y))/d(x, y)$. *Then* T *has a unique fixed point.*

The Theorem B is a generalized version of Theorem 4 of [14] which was stated as one of the best fixed point theorem by Rhoades in [14].

On the other hand, in [7] we introduced a $D_r(\alpha, \beta)$ class of multi-valued mapping on metric spaces, and proved a fixed point theorem for multi-valued mappings of class $D_r(\alpha, \beta)$ with $\alpha + 2\beta < 1$ on complete metric spaces. A multi-valued mapping T is said to be class $D_r(\alpha, \beta)$ if $D_r(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot (D_r(x, Ty) + D_r(y, Tx))$ (see section 2 for the definition of $D_r(\cdot, \cdot)$). In this article we will prove two fixed point theorems for multi-valued mappings on complete metric spaces, which generalize Wong's result[15], Rhoades' Theorem 4 in [14] and our results in [7].

2. Main theorems

Let (X, d) be a metric space and $C(X)$ the family of all nonempty closed subsets of X. For A and B in $C(X)$, we define an extended real number $D_r(A, B)$ as follows $D_r(A, B) = \sup_{x \in A} d(x, B)$, where $d(x, B) = \inf_{c \in B} d(x, c)$. Similarly, we define $D_l(A, B)$ by $D_l(A, B) = D_r(B, A) = \sup_{x \in B} d(x, A)$. Then so called Hausdorff distance $D(A, B)$ between A and B in $C(X)$ is obviously defined by $D(A, B) = \mathrm{Max}\ \{D_r(A, B), D_l(A, B)\}$. We note that all of $D(A, B)$, $D_r(A, B)$ and $D_l(A, B)$ actually depend on the metric d on X.

It is clear that for any $x, y \in X$, and $A \in C(X)$ we have $d(x, y) = D_r(x, y) = D_l(x, y)$ and $D_r(x, A) = d(x, A)$. And, in general, $D_r(A, B) \neq D_r(B, A)$ for $A, B \in C(X)$. So, $D_r(\cdot, \cdot)$ is not a metric on the space $C(X)$.

For $D_r(\cdot, \cdot)$ and $D_l(\cdot, \cdot)$ we have the following elementary properties which were proved in [7].

PROPOSITION 2.1. (i) $D_r(A) \geq 0$ *and* $D_l(A, B) \geq 0$,

(ii) $D_r(A, B) = 0$ *if and only if* $A \subset B$, *and* $D_l(A, B) = 0$ *if and only if* $B \subset A$,

(iii) (*triangle inequality*) $D_r(A, B) \leq D_r(A, C) + D_r(C, B)$, *and* $D_l(A, B) \leq D_l(A, C) + D_l(C, B)$, *for* $A, B, C \in C(X)$.

Let T be a mapping from X into $C(X)$. A point $x \in X$ is said to be a *fixed point* of T if $x \in Tx$. In this section we give a fixed point theorem for some kind of mappings from X into $C(X)$. We also prove the existance of common fixed points for two multi-valued mappings with some conditions. To prove these theorems we need an elementary lemma as follows.

LEMMA 2.2. *Let* $\{a_n\}$ *be a real sequence. If* $0 \leq a_{n+1} \leq a_n + \frac{1}{2^n}$, *for all* n, *then the sequence is convergent.*

Proof. $a_{n+1} \leq a_1 + \sum_{n=1}^{\infty} \frac{1}{2^n} = a_1 + 1$, by induction. Then $\{a_n\}$ is a bounded sequence. Suppose that there are two subsequences $\{a_{n_p}\}$, and $\{a_{n_q}\}$ which converge to a and b, respectively. Suppose $a > b$. Let $\varepsilon = \frac{1}{3}(a-b)$. Then there is an integer N such that $|a_{n_p} - a| < \varepsilon$, $|a_{n_q} - b| < \varepsilon$, for all $n_p, n_q \geq N$, and $\sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon$. Then $a - b \leq |a_{n_p} - a| + (a_{n_p} - a_{n_q}) + |a_{n_q} - b| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon = a - b$, when $n_p \geq n_q \geq N$. This contradiction implies $a = b$. Thus $\lim_{n \to \infty} a_n$ exists.

We can now deal with our main result in this paper which generalizes Wong's result [15], Rhoades' Theorem 4 in [14], and our result in [7] obviously.

THEOREM 2.3. *Let* T *be a mapping of a complete metric space* (X, d) *into* $C(X)$ *where* $C(X)$ *is the familly of all closed subsets of* X. *Suppose that there exist functions* $\alpha_i, i = 1, 2, 3$, *of* $(0, \infty)$ *into* $[0, \infty)$ *such that*

(i) *each* α_i *are continuous,*

(ii) $\alpha_1(t) + 2\cdot\alpha_2(t) + 2\cdot\alpha_3(t) < t$, *for* $t > 0$,

(iii) $\lim_{n \to 0+} \frac{\alpha_1(t) + \alpha_2(t) + \alpha_3(t)}{t - \alpha_2(t) - \alpha_3(t)} = k < 1$,

(iv) *for any* $x, y \in X$, $D_r(Tx, Ty) \leq$
$a_1 \cdot d(x, y) + a_2 D_r(x, Tx) + a_2 \cdot D_r(y, Ty) + a_3 \cdot D_r(x, Ty) + a_3 \cdot D_r(y, Tx)$, *where* $a_i = \alpha_i(d(x, y))/d(x, y)$, $i = 1, 2, 3$. *Then* T *has at least a fixed point.*

Proof. Let x_0 be an arbitrary point in X. Pick any point x_1 in Tx_0.

We can assume $x_1 \neq x_0$ because if $x_1 = x_0$ then x_0 is a fixed point of T. Then we can choose a $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq D_r(Tx_0, Tx_1) + \frac{1}{2}(1-a_2-a_3)$$
$$\leq a_1 \cdot d(x_0, x_1) + a_2 \cdot (D_r(x_0, Tx_0) + D_r(x_1, Tx_1)) + a_3 \cdot (D_r(x_0, Tx_1) + D_r(x_1, Tx_0)) + \frac{1}{2}(1-a_2-a_3)$$
$$\leq a_1 \cdot d(x_0, x_1) + a_2 \cdot (d(x_0, x_1) + d(x_1, x_2)) + a_3 \cdot d(x_0, x_2) + \frac{1}{2}(1-a_2-a_3)$$
$$\leq (a_1+a_2+a_3) \cdot d(x_0, x_1) + (a_2+a_3) \cdot d(x_1, x_2) + \frac{1}{2}(1-a_2-a_3).$$ That is

(2.1) $$d(x_1, x_2) \leq \frac{a_1+a_2+a_3}{1-a_2-a_3} \cdot d(x_0, x_1) + \frac{1}{2}.$$

Denote $b_0 = d(x_0, x_1)$, $b_1 = d(x_1, x_2)$. By the definition of a_i, (2.1) reduces to

(2.2) $$b_1 \leq \frac{\alpha_1(b_0) + \alpha_2(b_0) + \alpha_3(b_0)}{b_0 - \alpha_2(b_0) - \alpha_3(b_0)} \cdot b_0 + \frac{1}{2}.$$

By induction, we can pick $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ for all $n = 2, 3, \cdots$, shch that

(2.3) $$d(x_n, x_{n+1}) \leq D_r(Tx_{n-1}, Tx_n) + \frac{1}{2^n} \cdot (1-a_2-a_3).$$

By the same argument as before, (2.3) implies that

(2.4) $$b_n \leq \frac{\alpha_1(b_{n-1}) + \alpha_2(b_{n-1}) + \alpha_2(b_{n-1})}{b_{n-1} - \alpha_2(b_{n-1}) - \alpha_3(b_{n-1})} \cdot b_{n-1} + \frac{1}{2^n}.$$

(2.5) Thus $b_n \leq b_{n-1} + \frac{1}{2^n}$, for all n.

Then the sequence $\{b_n\}$ is convergent by Lemma 2.2. Denote the limit of $\{b_n\}$ by b. Taking limit as $n \to \infty$ in (2.4) we have

(2.6) $$b \leq \frac{\alpha_1(b) + \alpha_2(b) + \alpha_3(b)}{b - \alpha_2(b) - \alpha_3(b)} \cdot b,$$

since α_i are continuous. From (ii), b must be zero, i.e., $\lim_{n \to \infty} b_n = 0$. Then there are an integer N and a positive number $k_1 < 1$ such that

$\frac{\alpha_1(b_n)+\alpha_2(b_n)+\alpha_3(b_n)}{b_n-\alpha_2(b_n)-\alpha_3(b_n)} \leq k_1$ for $n \geq N$ by (iii). Hence we have

$$(2.7) \qquad b_{n+1} \leq k_1 \cdot b_n + \frac{1}{2^n}, \quad \text{for all } n \geq N.$$

Then $\{x_n\}$ is a Cauchy sequence in X and there exists a z in X such that $x_n \to z$ as $n \to \infty$ by the completeness of X. We shall show that z is a fixed point of T.

The inequality

$$\begin{aligned} D_r(z, Tz) &\leq d(z, x_n) + D_r(x_n, Tx_n) + D_r(Tx_n, Tz) \\ &\leq d(z, x_n) + d(x_n, x_{n+1}) + a_1 \cdot d(z, x_n) + a_2 \cdot D_r(x_n, Tx_n) + a_2 \cdot D_r(z, Tz) \\ &\quad + a_3 \cdot D_r(x_n, Tz) + a_3 \cdot D_r(z, Tx_n) \\ &\leq d(z, x_n) + b_n + a_1 \cdot d(z, x_n) + a_2 \cdot b_n + a_2 \cdot D_r(z, Tz) + a_3 \cdot d(x_n, z) \\ &\quad + a_3 \cdot d(z, Tz) + a_3 \cdot d(z, x_n) + a_3 \cdot b_n \end{aligned}$$

implies that

$$D_r(z, Tz) \leq \frac{1+a_1+2a_3}{1-a_2-a_3} d(x_n, z) + \frac{1+a_2+a_3}{1-a_2-a_3} b_n.$$

Let $c_n = d(x_n, z)$. Then $D_r(z, Tz) \leq A \cdot d(x_n, z) + B \cdot b_n$, where

$A = \frac{1+a_1+2a_3}{1-a_2-a_3} = \frac{c_n+\alpha_1(c_n)+2\alpha_3(c_n)}{c_n-\alpha_2(c_n)-\alpha_3(c_n)}$, and

$B = \frac{1+a_2+a_3}{1-a_2-a_3} = \frac{c_n+\alpha_2(c_n)+\alpha_3(c_n)}{c_n-\alpha_2(c_n)-\alpha_3(c_n)}$. Since

$$A = 1 + \frac{\alpha_1(c_n)+\alpha_2(c_n)+\alpha_3(c_n)}{c_n-\alpha_2(c_n)-\alpha_3(c_n)} + \frac{2\alpha_3(c_n)}{c_n-\alpha_2(c_n)-\alpha_3(c_n)}$$

$\leq 1+1+2=4$, $B = 1 + \frac{2\alpha_2(c_n)+2\alpha_3(c_n)}{c_n-\alpha_2(c_n)-\alpha_3(c_n)} \leq 3$, $b_n \to 0$, and $d(x_n, z) \to 0$ as $n \to \infty$, we have $D_r(z, Tz) = 0$ which implies $z \in Tz$.

We now prove a theorem on common fixed points of two multi-valued mappings S and T. Two multi-calued mappings S and T are said to have a common fixed point in X if there exists a point $x \in X$ such that $x \in Sx$ and $x \in Tx$, simultaneously.

THEOREM 2.4. *Let S, T bet two mappings from X into $C(X)$. Suppose that there exist functions α_i, $i=1, 2, 3$, of $(0, \infty)$ into $[0, \infty)$ such that*

(i) *each α_i are continuous,*

(ii) $\alpha_1(t) + 2\alpha_2(t) + 2\alpha_3(t) < t$,

(iii) $\lim_{t \to 0+} \frac{\alpha_1(t)+\alpha_2(t)+\alpha_3(t)}{t-\alpha_2(t)-\alpha_3(t)} = k < 1$,

(iv) *for any $x, y \in X$ we have*

$$D_r(Sx, Ty) \leq a_1 d(x, y) + a_2 D_r(x, Sx) + a_2 D_r(y, Ty) + a_3 D_r(x, Ty) + a_3 D_r(y, Sx)$$

$$D_r(Tx, Sy) \leq a_1 d(x, y) + a_2 D_r(x, Tx) + a_2 D_r(y, Sy) + a_3 D_r(x, Sy) + a_3 D_r(y, Tx)$$

where $a_i = \alpha_i(d(x, y))/d(x, y)$. *Then S and T have at least a common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Pick any point x_1 in Tx_0. Then choose a $x_2 \in Sx_1$ such that

$$\begin{aligned} d(x_2, x_1) &\leq D_r(Sx_1, Tx_0) + \frac{1}{2}(1-a_2-a_3) \\ &\leq a_1 \cdot d(x_1, x_0) + a_2 \cdot D_r(x_1, Sx_1) + a_2 \cdot D_r(x_0, Tx_0) + a_3 \cdot D_r(x_1, Tx_0) \\ &\qquad + a_3 D_r(x_0, Sx_1) + \frac{1}{2}(1-a_2-a_3) \\ &\leq a_1 \cdot d(x_0, x_1) + a_2 \cdot d(x_1, x_2) + a_2 \cdot d(x_0, x_1) + a_3 \cdot d(x_0, x_2) + \frac{1}{2}(1-a_2-a_3) \\ &\leq a_1 \cdot d(x_0, x_1) + a_2 \cdot d(x_1, x_2) + a_2 \cdot d(x_0, x_1) + a_3 \cdot d(x_0, x_1) + a_3 \cdot d(x_1, x_2) \\ &\qquad + \frac{1}{2}(1-a_2-a_3). \end{aligned}$$

(2.8) Thus $d(x_1, x_2) \leq \frac{a_1+a_2+a_3}{1-a_2-a_3} \cdot d(x_0, x_1) + \frac{1}{2}$.

(2.9) That is, $b_1 \leq \frac{\alpha_1(b_0)+\alpha_2(b_0)+\alpha_3(b_0)}{b-\alpha_2(b_0)-\alpha_3(b_0)} \cdot b_0 + \frac{1}{2}$,
where $b_i = d(x_i, x_{i+1})\, i=0,\ 1$.

We can also pick a $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq \frac{a_1+a_2+a_3}{1-a_2-a_3} \cdot d(x_1, x_2) + \frac{1}{2^2} \tag{2.10}$$

That is

$$b_2 \leq \frac{\alpha_1(b_1)+\alpha_2(b_1)+\alpha_3(b_1)}{b_1-\alpha_2(b_1)-\alpha_3(b_1)} b_1 + \frac{1}{4}, \text{ where } b_2 = d(x_2, x_3). \tag{2.11}$$

By induction, we can pick $x_{2n} \in Sx_{2n-1}$, $x_{2n+1} \in Tx_{2n}$, such that

$$b_{n+1} \leq \frac{\alpha_1(b_n)+\alpha_2(b_n)+\alpha_3(b_n)}{b_n-\alpha_2(b_n)-\alpha_3(b_n)} \cdot b_n + \frac{1}{2^n}, \tag{2.12}$$

for all n, where $b_n = d(x_n, x_{n+1})$.

By the same argument as we have done in the proof of Theorem 2.3, we can prove that $\lim_{n\to 0} b_n=0$, $\{x_n\}$ is a Cauchy sequence in X, and $\lim_{n\to\infty} x_n=z$, since X is complete.

We prove that z is a common fixed point of S and T. First of all, we prove $z\in Tz$.

$$\begin{aligned} D_r(z, Tz) &\leq d(z, x_{2n+1})+D_r(x_{2n+1}, Sx_{2n+1})+D_r(Sx_{2n+1}, Tz) \\ &\leq d(z, x_{2n+1})+d(x_{2n+1}, x_{2n+2})+a_1\cdot d(x_{2n+1}, z)+a_2\cdot D_r(x_{2n+1}, Sx_{2n+1}) \\ &\qquad +a_2\cdot D_r(z, Tz)+a_3\cdot D_r(z, Sx_{2n+1})+a_3\cdot D_r(x_{2n+1}, Tz) \\ &\leq d(z, x_{2n+1})+d(x_{2n+1}, x_{2n+2})+a_1\cdot d(x_{2n+1}, z)+a_2\cdot d(x_{2n+1}, x_{2n+2}) \\ &\qquad +a_2\cdot D_r(z, Tz)+a_3\cdot d(z, x_{2n+1})+a_3\cdot d(x_{2n+1}, x_{2n+2}) \\ &\qquad +a_3\cdot d(x_{2n+1}, z)+d(z, Tz). \end{aligned}$$

Thus $D_r(z, Tz)\leq\dfrac{1+a_1+2a_3}{1-a_2-a_3}\cdot d(z, x_{2n+1})+\dfrac{1+a_2+a_3}{1-a_2-a_3}\cdot d(x_{2n+1}, x_{2n+2})$.

By the same method used in the proof of Theorem 2.3, we conclude $z\in Tz$. Similarly, we can prove $z\in Sz$. Therefore z is a common fixed point of S and T.

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