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ON RECURRENT HYPERSURFACES OF A REAL SPACE FORM*

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1. Introduction

For a given Riemannian manifold M, if M admits a single submanifold with a special property, what can we say about the ambient space? It seems to be natural and interesting to find out the implication on the ambient space from a certain submaifold. Since this problem was initiated by Chen and Nagano [1], it is seen that there are some studies in this direction. In particular, one of the authors [4] proved recently the following.

THEOREM A. *The only irreducible symmetric space which admits a connected recurrent hypersurface with at most two principal curvatures, each of which multiplicity is not smaller than* 3, *are a sphere, a real projective space and their non compact duals.*

On the other hand, it is seen in [2] that a complete simply connected recurrent Riemannian manifold is either a symmetric space or a direct product of the Euclidean space R^{n-2} and a 2-dimensional Riemannian manifold. The purpose of this article is to consider "what can we say about recurrent hypersurfaces of a real space form?" Concerning with this standpoint we prove the following.

THEORM. *Let M be a complete hypersurface of an $(n+1)$-dimensional sphere $S^{n+1}(c)$. If M is recurrent and if $n\geqq 4$, then M is isometric to the sphere $S^n(c_1)$ $(c_1\geqq c)$ or the product of two spheres $S^p(c_1)\times S^{n-p}(c_2)$, where $1/c_1+1/c_2=1/c$.*

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2. Preliminaries

Let (M', g') be an $(n+1)$-dimensional connected Riemannian manifold of constant sectional curvature c, which is denoted by $M^{n+1}(c)$ and is called a real space form. Let (M, g) be a hypersurface of M'. We choose a local orthonormal frame field $\{E_A\}=\{E_1, \cdots, E_{n+1}\}$ on a neighborhood of M' in such a way that, restricted to $M, E_1, \cdots, E_n$ are tangent to M and hence the other is normal to M. Here and in the sequal, the following convention on the range of indices are used throughout this paper, unless otherwise stated;

$$A, B, \cdots=1, \cdots n, n+1,$$
$$i, j, \cdots=1, \cdots, n.$$

With respect to the frame field, let $\{\omega_A\}=\{\omega_j, \omega_{n+1}\}$ be its dual frame field. Then we have the structure eguations for M' :

$$\begin{cases} d\omega_A+\Sigma\omega_{AB}\wedge\omega_B=0, \\ d\omega_{AB}+\Sigma\omega_{AC}\wedge\omega_{CB}=c\omega_A\wedge\omega_B, \end{cases} \tag{2.1}$$

where ω_{AB} denotes the connection forms on M'.

Restricting these forms to the hypersurface M, we have

$$\omega_{n+1}=0 \tag{2.2}$$

and the Riemannian metric of M induced from the Riemannian metric g' on the ambient space is given by $g=2\Sigma\omega_j\otimes\omega_j$. Therefore $\{E_j\}$ is a local orthonormal frame field with respect to this metric and $\{\omega_j\}$ is a local dual field relative to $\{E_j\}$. It follows from the Cartan lemma that the exterior derivative of the first equation of (2.1) gives rise to

$$\omega_{n+1i}=\Sigma h_{ij}\omega_j, \quad h_{ij}=h_{ji}. \tag{2.3}$$

The guadratic form $\Sigma h_{ij}\omega_i\otimes\omega_j\otimes E_{n+1}$ with values in the normal bundle is called the second fundamental form of the hypersurface M.

From the structure equations for M' it follows that the structure equations for M are similarly given by

$$\begin{cases} d\omega_i+\Sigma\omega_{ij}\wedge\omega_j=0, \\ d\omega_{ij}+\Sigma\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}, \\ \qquad \Omega_{ij}=-\frac{1}{2}\Sigma R_{ijkl}\omega_k\wedge\omega_l, \end{cases} \tag{2.4}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R). Form the equations obtained above it follows that we have the Gauss equation

$$R_{ijkl}=c(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+h_{il}h_{jk}-h_{ik}h_{jl}. \tag{2.5}$$

Now, the components h_{ijk} of the covariant derivative of the second fundamental form of M are given by

$$\Sigma h_{ijk}\omega_k=dh_{ij}-\Sigma(h_{kj}\omega_{ki}+h_{ik}\omega_{kj}).$$

Then, substituting dh_{ij} in this definition into the exterior derivative of (2.3), we have the Codazzi equation

$$h_{ijk}=h_{ikj}. \tag{2.6}$$

The components $R_{ijkl,m}$ of the covariant derivative of R are defined by

$$\Sigma R_{ijkl,m}\omega_m=dR_{ijkl}-\Sigma(R_{mjkl}\omega_{mi}+R_{imkl}\omega_{mj}+R_{ijml}\omega_{mk}+R_{ijkm}\omega_{ml}).$$

The components $R_{ijkl,mn}$ of the covariant derivative of $R_{ijkl,m}$ are defined by

$$\Sigma R_{ijkl,mn}\omega_n=dR_{ijkl,m}-\Sigma(R_{njkl,m}\omega_{ni}+R_{inkl,m}\omega_{nj}+R_{ijnl,m}\omega_{nk}+R_{ijkn,m}\omega_{nl}+R_{ijkl,n}\omega_{nm}).$$

The Riemannian manifold is said to be *recurrent*, if there exists a 1-form η such that

$$R_{ijkl,m}=\eta_m R_{ijkl}. \tag{2.7}$$

REMARK. For a recurrent Riemannian manifold, the Riemannian curvature tensor R satisfies

$$R(X,Y)R=0 \tag{2.8}$$

for all tangent vectors X and Y, where the endomorphism $R(X,Y)$ operates on R as a derivation of the tensor algebra at each point. In fact, it follows from (2.7) that we have

$$|R|_m^2=2|R|^2\eta_m,$$

where $|R|$ denotes the norm of the Riemannian curvature tensor R. Let U be the set consisting of points x at which $|R|(x)\neq 0$. Then there is a smooth function ϕ on U such that $d\phi=\eta$, which is called the *characteristic function.* The function ϕ is given by $\log|R|$. It implies that

$$R_{ijkl,mn}=(\phi_{mn}+\phi_m\phi_n)R_{ijkl}=R_{ijkl,nm},$$

which means that the equation (2.8) is true on U, and therefore on the whole manifold.

3. Proof of the theorem.

This section is concerned with principal curvatures of recurrent hypersurfaces of a real space form. Let M be a recurrent hypersurface of a real space form $M'=M^{n+1}(c)$ $(n\geqq 3)$. Since the recurrent hypersurface M satisfies (2.8) and hence it satisfies

$$R(X, Y)S=0 \text{ for all vectors } X \text{ and } Y,$$

where S is the Ricci tensor, it is seen in Mogi and Nakagawa [3] and Tanno [5] that if $c\neq 0$, then there exist at most two distinct principal curvatures at each point x of M and the type number $t(x)$ is equal to $0, 1$ or n, and if $c=0$, then there exist at most three distinct principal curvatures at each point of M, one of which is zero.

On the other hand, if M is locally symmetric, then its Ricci tensor is parallel and therefore it is seen in [3] that M is an isoparametric hypersurface, which is completely classified.

Accordingly, in order to prove the theorem mentioned in the introduction, we may assume that M is not locally symmetric. Then there exists a neighborhood V contained in U on which $d\phi$ does not vanish.

Now, taking account of the second Bianchi formula, the definition (2.7) and the property of the Riemannian curvature tensor, we obtain

$$\phi_k R_{ijkl}+\phi_i R_{jkkl}+\phi_j R_{kikl}=0. \tag{3.1}$$

Since the matrix (h_{ij}) is symmetric, it can be diagonalizable and we can choose a local field of orthonormal frames on M in such a way that $h_{ij}=\lambda_i\delta_{ij}$, where each eigenvalue λ_i is called a principal curvature of M. Then the Gauss equation (2.5) and (3.1) yield

(3. 2) $\phi_h(c+\lambda_i\lambda_j)(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})+\phi_i(c+\lambda_j\lambda_k)(\delta_{jl}\delta_{hk}-\delta_{jk}\delta_{hl})$
$+\phi_j(c+\lambda_k\lambda_i)(\delta_{hl}\delta_{ik}-\delta_{hk}\delta_{il})=0.$

Put $l=h$, $k=j$ and suppose that indices i, j and h are mutually distinct, we obtain

$$\phi_i(c+\lambda_j\lambda_h)=0. \tag{3.3}$$

Since the gradient of ϕ does not vanish identically on V, we may suppose without loss of generality that there is a neighborhood V_1 contained in V, on which ϕ_1 has no zero points by changing suitably the order of the orthonormal frames. It follows that

$$c+\lambda_i\lambda_j=0 \text{ for } i, j>1, \ i\neq j, \tag{3.4}$$

which implies that distinct principal curvatures $\lambda_j (j\geqq 2)$ are at most two on V, say μ and σ.

Assume that $n\geqq 4$. The multiplicities of distinct principal curvatures μ and σ $(c+\mu\sigma=0)$ are then investigated. Let p (resp. q) be a multiplicity of the principal curvature μ (resp. σ). We suppose that $p, q\geqq 2$. Then the equation (3. 3) means that

$$\mu\sigma=\mu^2=\sigma^2=-c,$$

which implies that $c=0$ and $\mu=\sigma=0$, a contradiction. Next we suppose that $p=1$ and $q\geqq 2$. In this case we have

$$\mu\sigma=\sigma^2=-c,$$

which implies that $c=0$ and $\sigma=0$. Thus the matrix h is expressed by

$$h=\begin{bmatrix} \lambda & & & \\ & \mu & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{bmatrix} \text{ or } h=\begin{bmatrix} \lambda & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{bmatrix}, \quad \mu\neq 0.$$

According as $c=0$ or $c<0$.
Thus we find

LEMMA 1. *Let M be a recurrent hypersurface of M'. If M is not locally symmetric and if $n\geqq 4$, then the constant curvature c is non-positive.*

By Lemma 1 and the classification of isoparametric hypersurfaces in a sphere the theorem is proved.

REMARK. Under the assumption of Theorem A the ambient space is a sphere or a real projective space.

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