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ON CONFORMAL DIFFEOMORPHISM BETWEEN ALMOST HERMITIAN MANIFOLDS AND KAEHLERIAN MANIFOLDS

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1. Introduction

Under a non-homothetic conformal diffeomorphism of an almost Hermitian manifold M onto a Kaehlerian manifold M^* of real dimension >4, M is a non-Kaehlerian Hermitian manifold. As a subclass of almost Hermitian manifolds, RK-manifolds have interesting geometrical properties and many geometers are concerned with them (see [1], [6], [7]).

A conformal diffeomorphism f of a Riemannian manifold (M, g) onto a Riemannian manifold (M^*, g^*) is characterized by the metric change

$$f^*g^*=\rho^{-2}g, \tag{1.1}$$

where ρ is a positive scalar field on M. The purpose of this paper is to prove

THEOREM 1. *If there is a non-homothetic conformal diffeomorphism f of an almost Hermitian manifold M onto a Kaehlerian manifold M^* of real dimension>4, then f preserves the holomorphic sectional curvatures H of M and H^* of M^*, that is, $\rho^2H=H^*$ and at least one of the scalar curvatures of M and M^* are not constant.*

THEOREM 2. *Let f be a conformal diffeomorphism of an almost Hermitian manifold M onto a Kaehlerian manifold M^* of real dimension >4. Then M is a RK-manifold if an only if the characteristic scalar field ρ satisfies the equation*

$$\nabla_X d\rho=cX,$$

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for any vector field X on M, where c is a positive constant. In the case when M is complete, both M and M^* are locally Euclidean.

2. Preliminaries

Let M and M^* be $2n$-dimensional Riemannian manifolds with Riemannian metrics g and g^* respectively. Under the conformal diffeomorphism f characterized by (1.1), we shall denote the gradient vector field of $\log \rho$ by U, that is,

$$(2.1) \qquad g(U, X) = X \log \rho$$

for any vector field X on M. The Riemannian connections with respect to g and g^* will be denoted by ∇ and ∇^* respectively. Then it follows from (1.1) and (2.1) that

$$(2.2) \qquad \nabla^*_X Y - \nabla_X Y = g(X, Y)U - g(U, Y)X - g(U, X)Y$$

for any vector fields X and Y on M. From now on, quantities of M^* associated to that of M will be denoted by asterisking.

A bilinear 2-form P on M defined by

$$(2.3) \qquad P(X, Y) = (XY - \nabla_X Y)\log \rho + g(U, X)g(U, Y) - \frac{1}{2}|U|^2 g(X, Y)$$

is symmetric for any vector fields X and Y on M, where $|U|$ is the magnitude of U. A linear transformation Q of M associated to P is given by

$$(2.4) \qquad g(QX, Y) = P(X, Y).$$

The curvature tensor, the Ricci tensor and the scalar curvature of M will be denoted by K, Ric and κ respectively. Then using (2.2), (2.3) and (2.4), we have the transformation formulas ([5])

$$(2.5) \qquad K^*(X, Y)Z = K(X, Y)Z + P(Y, Z)X - P(X, Z)Y + g(Y, Z)QX - g(X, Z)QY,$$

$$(2.6) \qquad \mathrm{Ric}^*(X, Y) = \mathrm{Ric}(X, Y) + 2(n-1)P(X, Y) + \sum_i P(E_i, E_i)g(X, Y),$$

$$(2.7) \qquad \rho^{-2}\kappa^* = \kappa + 2(2n-1)\sum_i P(E_i, E_i),$$

for any vector fields X, Y and Z on M, where $\{E_i\}$ is any orthonormal frame for M.

Now assume that M is an almost Hermitian manifold with the structure (g, J) and M^* is a Kaehlerian manifold with the structure (g^*, J^*). The structure (g, J) satisfies

$$J^2=-I,\ g(JX,\ JY)=g(X, Y),$$

for any vector fields X and Y on M, and (g^*, J^*) does

$$(2.8)\quad J^{*2}=-I,\ g^*(J^*X^*,\ J^*Y^*)=g^*(X^*, Y^*),\ \nabla^*J^*=0$$

for any vector fields X^* and Y^* on M^*. Since the conformal diffeomorphism $f: M\to M^*$ is holomorphic, f preserves the complex structure, that is,

$$f^*J^*=J^*f_*.$$

Therefore it follows from (1.1) and the third of (2.5) that

$$(2.9)\quad (\nabla_X J)Y=g(U, JY)X-g(U, Y)JX+g(X, Y)JU-g(X, JY)U.$$

The Nijenhuis tensor N of M is given by

$$N(X, Y)=[JX, JY]-J[X, JY]-J[JX, Y]-[X, Y].$$

It is easily verified that N vanishes identically on M by virtue of (2.9). Thus we can state

LEMMA 2.1. *If there is a conformal diffeomorphism f of an almost Hermitian manifold M onto a Kaehlerian manifold M^*, then M is a Hermitian manifold.*

An almost Hermitian manifold M is called a *RK-manifold* if the curvature tensor field K of M satisfies

$$(2.10)\qquad g(K(JX, JY)JZ, JW)=g(K(X, Y)Z, W).$$

The geometry of the RK-manifolds has been studied by L. Vanheke ([6]) and L. Vanheke and K. Yano ([7]).

3. Proof of the Theorems

First of all, differentiating (2.7) covariantly along M, we have

$$(3.1)\quad \begin{aligned} &K(X, Y, JZ, JW)-K(X, Y, Z, W)\\ &=g(X, W)P(Y, Z)-g(Y, W)P(X, Z)\\ &\quad -g(X, JW)P(Y, JZ)+g(Y, JW)P(X, JZ)\\ &\quad +P(X, W)g(Y, Z)-P(Y, W)g(X, Z)\\ &\quad -P(X, JW)g(Y, JZ)+P(Y, JW)g(X, JZ) \end{aligned}$$

for any vector fields X, Y, Z and W on M. If we apply the first Bianchi identity to (3.1), then we obtain

$$(3.2)\quad \begin{aligned} &g(X, JW)\{P(Z, JY)-P(Y, JZ)\}+2P(X, JW)g(Z, JY)\\ &+g(Y, JW)\{P(X, JZ)-P(Z, JX)\}+2P(Y, JW)g(X, JZ)\\ &+g(Z, JW)\{P(Y, JX)-P(X, JY)\}+2P(Z, JW)g(Y, JX)\\ &=0 \end{aligned}$$

Let $\{E_1, \ldots, E_{2n}\}$ be any orthonormal frame for M. Putting $X=JE_i$ and $W=E_i$ into (3.2) and summing over i, we have

$$(3.3)\quad (n-2)\{P(Z, JY)-P(Y, JZ)\}+\sum_i P(JE_i, JE_i)g(Z, JY)=0.$$

Putting $Y=E_i$ and $Z=JE_i$ again into (3.3) and taking account of the relation $\sum_i P(JE, JE_i)=\sum_i P(E_i, E_i)$, we also have

$$(3.4)\quad \sum_i P(E_i, E_i)=0.$$

Then it follows from (3.3) and (3.4) that

$$(3.5)\quad P(X, JY)=P(Y, JX).$$

The holomorphic sectional curvature $H(X, JX)$ of any holomorphic section $\rho(X, JX)$ in an almost Hermition manifold M is given by

$$(3.6)\quad H(X, JX)=-\frac{g(K(X, JX)X, JX)}{\{g(X, X)\}^2}$$

for any vector fields X on M. Since the equations (2.5) and (3.5) lead to

$$K^*(X, JX, X, JX)=\rho^{-2}K(X, JX, X, JX),$$

we have

$$H^*(X, JX)=\rho^2 H(X, JX)$$

which shows that f preserves the holomorphic sectional curvatures. Moreover, comparing (2.7) with (3.4), we have

$$\kappa^*=\rho^2\kappa,$$

which means that one of the scalar curvatures κ^* and κ is not a constant. This completes the proof of Theorme 1.

To prove Theorem 2, if we replace X and Y in (3.1) with JX and JY respectively, then we have

$$\begin{aligned}(3.7)\qquad & K(JX, JY, JZ, JW)-K(JX, JY, Z, W)\\ &=g(JX, W)P(JY, Z)-g(JY, W)P(JX, Z)\\ &\quad -g(X, W)P(JY, JZ)+g(Y, W)P(JX, JZ)\\ &\quad +P(JX, W)g(JY, Z)-P(JY, W)g(JX, Z)\\ &\quad -P(JX, JW)g(Y, Z)+P(JY, JW)g(X, Z).\end{aligned}$$

By a substitution of (3.1) into (3.7), we get

$$\begin{aligned}(3.8)\qquad & K(JX, JY, JZ, JW)-K(X, Y, Z, W)\\ &=g(X, Z)\{P(JY, JW)-P(Y, W)\}\\ &\quad -g(Y, Z)\{P(JX, JW)-P(X, W)\}\\ &\quad -g(X, W)\{P(JY, JZ)-P(Y, Z)\}\\ &\quad +g(Y, W)\{P(JX, JZ)-P(X, Z)\}.\end{aligned}$$

Since M is a RK-manifold, we can easily obtain from (2.10) and (3, 8) that

$$(3.9)\qquad P(JX, JY)=P(X, Y).$$

Therefore it follows from (3.5) and (3.9) that $P(X, Y)=0$, that is,

$$(3.10)\qquad (XY-\nabla_X Y)\log\rho=-g(U, X)g(U, Y)+\frac{1}{2}|U|^2 g(X, Y).$$

We see that the equation (3.10) is equivalent to

$$(3.11)\qquad \nabla_X d\rho=\frac{1}{2}\,\frac{|\mathrm{grad}\ \rho|^2}{\rho}X.$$

In terms of the canonical coordinate $\left\{\dfrac{\partial}{\partial x^\lambda}\right\}$ of M, (3.11) is expressed by

$$\nabla_j \rho_i = \frac{1}{2} \frac{|\text{grad } \rho|^2}{\rho} g_{ij}.$$

Applying $2\rho^i$ to the above equation, we have $|\text{grad } \rho|^2 = c' \rho$, where c' is a constant. Thus (3.11) is reduced to

$$\nabla_X d\rho = cX, \tag{3.12}$$

which shows that ρ is a concircular scalar field (see [1]).

Conversely if ρ satisfies the equation (3.12), we can easily show that M is a RK-manifold.

The remaining part of the theorem 2 follows from the results due to Y. Tashiro ([3], [4]). If a complete Riemannian manifold of dimension >2 admits a concircular scalar field given by (3.12), M is locally Euclidean

REMARK. The Bochner curvature tensor B^* of a Kaehlerian manifold M^* is givn by

$$\begin{aligned} B^*(X^*, Y^*)Z^* = {} & K^*(X^*, Y^*)Z^* + g^*(Y^*, Z^*)C^*X^* \\ & - g^*(X^*, Z^*)C^*Y^* - L^*(X^*, Z^*)Y^* \\ & + L^*(Y^*, Z^*)X^* + g^*(J^*Y^*, Z^*)C^*J^*X^* \\ & - L^*(J^*X^*, Z^*)J^*Y^* + L^*(J^*Y^*, Z^*)J^*X^* \\ & - g^*(J^*X^*, Z^*)C^*J^*Y^* - 2L^*(J^*X^*, Y^*)J^*Z^* \\ & - 2g^*(J^*X^*, Y^*)C^*J^*Z^* \end{aligned} \tag{3.13}$$

for any vector fields X^*, Y^* and Z^* on M^*, where we have put

$$\begin{aligned} L^*(X^*, Y^*) = {} & -\frac{1}{2(n+2)} \text{Ric}^*(X^*, Y^*) \\ & + \frac{\kappa^*}{8(n+1)(n+2)} g^*(X^*, Y^*) \end{aligned} \tag{3.14}$$

and C^* is its corresponding transformation defined by

$$g^*(C^*X^*, Y^*) = L^*(X^*, Y^*).$$

Under the conformal diffeomorphism f of M onto M^*, it follows from (2.6), (2.7) and (3.4) that

$$L^*(X, Y) = L(X, Y) - \frac{n-1}{n+2} P(X, Y) \tag{3.15}$$

for any vector fields X and Y on M, where we have put

$$L(X, Y)=-\frac{1}{2(n+2)}\mathrm{Ric}(X, Y)+\frac{\kappa}{8(n+1)(n+2)}g(X, Y).$$

And we shall denote by C the corresponding transformation to L. Substituting (2. 5) and (3. 15) into (3. 13), we obtain

$$\begin{aligned}(3.16)\qquad B^*(X, Y)Z=&B(X, Y)Z+\frac{3}{n+2}\{P(Y, Z)X\\ &-P(X, Z)Y+g(Y, Z)QX-g(X, Z)QY\}\\ &+\frac{n-1}{n+2}\{g(JY, Z)QJX-g(JX, Z)QJY\\ &+P(JX, Z)JY-P(JY, Z)JX-2P(JX, Y)JZ\\ &-2g(JX, Y)QJZ\}\end{aligned}$$

for any vector fields X, Y and Z on M, where

$$\begin{aligned}B(X, Y)Z=&K(X, Y)Z+g(Y, Z)CX-g(X, Z)CY-L(X, Z)Y\\ &+L(Y, Z)X+g(JY, Z)CJX-L(JX, Z)JY+L(JY, Z)JX\\ &-g(JX, Z)CJY-2L(JX, Y)JZ-2g(JX, Y)CJZ.\end{aligned}$$

Such B is called a *Bochner curvature tensor of an almost Hermitian manifold* (see [6]).

Suppose that f preserves the Bochner curvature tensors B and B^*, that is, $B=B^*$ on M, then we have

$$\begin{aligned}(3.17)\qquad 3\{&P(Y, Z)X-P(X, Z)Y+g(Y, Z)QX-g(X, Z)QY\\ &-(n-1)\{g(JY, Z)QJX-g(JX, Z)QJY\\ &+P(JX, Z)JY-P(JY, Z)JX-2P(JX, Y)JZ\\ &-2g(JX, Y)QJZ\}=0\end{aligned}$$

For any othonormal frame for M, putting $Y=JE_i$ and $Z=E_i$ into (3. 17) and summing over i, we also have

$$(3.18)\qquad P(X, Y)=0,$$

which shows that M is a RK-manifold. Conversely if M is a RK-manifold, we see that $B=B^*$ on M. Thus we can state

PROPOSITION 3. *Let f be a conformal diffeomorphism of an almost Hermitian manifold M onto a Kaehlerian manifold M^*. Then M is a*

RK-manifold if and only if f preserves the Bochner curvature tensors B of M and B^* of M^*.

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