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IRREDUCIBILITY OF FAMILIES OF COMPLEX PROJECTIVE CURVES

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1. Introduction

Throught this paper, base field is the field of complex numbers. We mean a curve by an algebraic curve in $\boldsymbol{P}^r$, which may be smooth or singular. The main object of our study is a parameter space of complex projective curves $I'_{d,g,r}$; the union of the irreducible components of the Hilbert scheme $\mathcal{H}_{d,g,r}$ whose general points correspond to smooth irreducible and nondegenerate curves.

In case $\rho(d,g,r)=g-(r+1)(g-d+r)\geq 0$, for a general curve C the dimenson of $W_d^r(C)$ is equal to $\rho(d,g,r)$ [8] and its general member is very ample [6]. Moreover it is irreducible for $\rho(d,g,r)>0$ [7]. Hence there may be a component of $I'_{d,g,r}$ whose general curve is the image curve of the morphism associated with the above series for $\rho(d,g,r)\geq 0$. In fact, the existence of such a component is stated in [9]. And it will be shown concretely in section 2.

J. Harris proposed the following conjecture in 1981, which seems to be the most prominent one concerning $I'_{d,g,r}$.

CONJECTURE. *$I'_{d,g,r}$ is irreducible for $\rho(d,g,r)\geq 0$.*

Recently in case $\rho(d,g,2)>0$, Arbarello and Zariski independently solved Severi's problem which is very similar to the above conjecture, that is, the variety of irreducible plane curves of degree d with δ nodes is irreducible. This in turn implies that the union of the components of $\mathcal{H}_{d,g,2}$ whose general points correspond reduced, irreducible, nondegenerate curves is irreducible for $\rho(d,g,2)>0$. On the one hand, we found out that $I'_{d,g,r}$ is reducible for $r=g-8$, $d=2g-8$. But such a counter

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example occurs only in higher dimensional space, as we will see in section 4. On the other hand, any abstract curve can be embedded in $\boldsymbol{P}^3$. Thus in these points of view, it is quite meaningful to investigate the irreducibility of $I'_{d,g,3}$.

It was shown in [9] that $I'_{d,g,3}$ is irreducible for $d>\frac{5}{4}g+1(\rho(d,g,3)>2g-8)$, while $\rho(d,g,3)\geq 0$ implies $d\geq\frac{3}{4}g+3$. Our main result states that $I'_{d,g,3}$ is irreducible for $d>g(\rho(d,g,3)>g-12)$. In fact we expect that $I'_{d,g,3}$ is irreducible in much wider range in the course of the proof of the result.

2. The irreducibility of $I'_{d,g,r}$

In order to look at our main object effectively, that is, a family of curves of genus g and degree d in $\boldsymbol{P}^r$ in some sense, we need to consider the following objects. Let $\mathscr{D}=g_d^r$ be a general subseries of a base point free simple series $|D|$ on an abstract curve $\widetilde{C}, F$ a frame in $V\subset H^0(C,\mathscr{O}(D))$ associated with $\mathscr{D}$. Then $\phi_{\mathscr{D}}$. which is defined by $\phi_{\mathscr{D}}(P)=(f_0(P),\dots,f_r(P))$, $P\in\widetilde{C}$, gives the birational morphism, where $F=(f_0,\dots,f_r)$. Conversely upon fixing a projective coordinate in $\boldsymbol{P}^r$, we can find such a triple $(\widetilde{C},\mathscr{D},F)$ by the normalization of C and the hyperplane section of C. Consequently Hilbert scheme $\mathscr{H}_{d,g,r}$ can be considered as the family of the triple $(\widetilde{C},\mathscr{D},F)$. Therefore we have to consider the pair $(\widetilde{C},\mathscr{D})$ over the general curves.

We denote by M_g the moduli space of the smooth curves of genus g and by $M_g{}^0$ the complement of the subvariety of M_g corresponding to curves with nontrivial automorphisms. As is well-known, there is a universal curve C over $M_g{}^0$, and the universal Jacobian $J_d\xrightarrow{p}M_g{}^0$ parameterizing the line bundles of degree d over $M_g{}^0$. It may be considered as parameterizing the complete linearserieses $(\mathscr{D},\widetilde{C})$ of degree d over $M_g{}^0$. We denote by $W_d{}^r$, $G_d{}^r$, respectively, the union of the line bundles L of J_d with $H^0(C,L)\geq r+1$ and the scheme parameterizing pairs $(g_d{}^r,\widetilde{C})$ over $M_g{}^0$. Then $W_d{}^r$ is in fact a closed subvariety of J_d. And we set

$$W_d{}^r(C)\ W_d{}^r\cap p^{-1}(C),\quad G_d{}^r(C)=G_d{}^r\cap p^{-1}(C)$$

where $p: G_d{}^r\to M_g{}^0$; since there will be no danger of ambiguity, we use the same letter p as above.

Then the object of our study $I'_{d,g,r}$ is really a PGL_{r+1}- bundle over the components of G_d^r whose general series $(g_d^r, \tilde{C})$ is very ample. Let G be the union of such a components of G_d^r. Then the irreducibility of G implies the irreducibility of $I'_{d,g,r}$. We will, in fact, show that G is irreducible in case $r=3$, $d>g$.

There is a natural morphism m of $I'_{d,g,r}$

$$m : I'_{d,g,r} \longrightarrow M_g$$

defined by $m((\tilde{C}, g_d^r, F)) = \tilde{C}$. A component S of $I'_{d,g,r}$ (or R of G_d^r) is said to be dominating M_g if $\overline{m(S)} = M_g (\overline{p(R)} = M_g)$.

In case $d \geq 2g-1$, for any curve C every divisor D of degree d has dimension $d-g$, the dimension of $W_d^r(C)$ is equal to g, a genral member of $W_d^r(C)$ is very ample for $r \leq d-g$. Consequently in this case $I'_{d,g,r}$ is empty if $r > d-g$, and irreducible of dimension $\lambda(d,g,r) = 4g-3+(r+1)(d-g+1)-1$ (this is the dominating component) if $r \leq d-g$.

On the other hand, we have the following useful theorem concerning G_d^r.

THEOREM 2. 1. *Every component of* G_d^r *has dimension at least* $3g-3+\rho(d,g,r)$. (See [2])

This implies that the dimension of any component of $I'_{d,g,r}$ is no less than $3g-3+\rho(d,g,r)+(r+1)^2-1 = \lambda(d,g,r)$. These facts are available for the study of our problem.

Accordingly we have the following result which is stated in [9].

THEOREM 2. 2. *In case* $\rho(d,g,r) \geq 0$, *there is a dominating component of* $I'_{d,g,r}$. *Moreover such a component is unique if* $\rho(d,g,r) > 0$.

Proof. If $\rho(d,g,r) \geq 0$, then dim $G_d^r(C) = \rho(d,g,r)$ and its general series is very ample on a general curve C [6]. Let S_0 be the closure in $\mathcal{H}_{d,g,r}$ of the locus of curves given by the general series over general curves. Then the closure of any other family of curves which does not dominate M_g, cannot properly contain S_0. Now a general series g_d^r of $G_d^r(C)$ over the general curves C generating S_0, is a complete series for $r \geq d-g$ and a subseries of the complete series $|D| = g_d^{d-g}$ for $r < d-g$. Let us consider the closure S in $\mathcal{H}_{d,g,r}$ of the locus of the curves given by a general subseries of a general $|D| = g_d^{\alpha}$ with $\alpha > r$ or $d-g$

respectively, in case $\rho(d, g, \alpha) \geq 0$. We need not investigate for α with $\rho(d, g, \alpha) < 0$, since in that case $W_d^\alpha(C)$ is empty on a general curve and we consider the dominating components. At first, we compute the dimension of S in case $r > d-g$. Then we put

$$\alpha = r+k, \quad k>0.$$

$$\begin{aligned}\dim S &= 3g-3+g-(\alpha+1)(g-d+\alpha)+(\alpha+1)(r+1)-1\\ &= 3g-3+g+(r+1)(d-g+1)+k(d-g+1)-k(r+1)-k^2-1\\ &< 3g-3+g+(r+1)(d-g+1)-1=\lambda(d, g, r).\end{aligned}$$

And in case that $r \leq d-g$, we set

$$\alpha = d-g+k, \quad k>0$$

$$\begin{aligned}\dim S &= 3g-3+g-(\alpha+1)(d-g+\alpha)+(\alpha+1)(r+1)-1\\ &= 3g-3+g+(r+1)(d-g+1)+k(r+1)-k(d-g+1)-k^2-1\\ &< \lambda(d, g, r).\end{aligned}$$

Hence S cannot be a component of $I'_{d,g,r}$. In particular it does not contain S_0. As a result, S_0 is the union of the dominating components of $I'_{d,g,r}$. In particular, in case $\rho(d, g, r) > 0$ $G_d^r(C)$ is irreducible on a general curve C [7] and so is S_0.

REMARK (1) Let G_0 be the union of the dominating components of the above G. Then G_0 is nonempty for $\rho(d, g, r) \geq 0$ and G_0 is irreducible for $\rho(d, g, r) > 0$. Note that G_0 is the closure in G_d^r of the locus of the pairs (g_d^r, C), which is a general very ample series on a general curve C.

(2) J. Harris' conjecture consequently implies that there is no component of $I'_{d,g,r}$ other than the unique dominating component S_0 in case $\rho(d, g, r) > 0$.

3. Irreducibility of $I'_{d,g,3}$

From now on, we only consider the case $\rho(d, g, 3) > 0$, that is, $d > \frac{3}{4}g+3$. Then a general member of G_0 in G_d^3 is a complete series g_d^3 for $d-g \leq 3$ and a general subseries g_d^3 of a complete series g_d^{d-g} for $d-g > 3$ on a general curve.

For $d > g+3$, assume that G' is a component of G whose general member is a subseries g_d^3 of a complete g_d^{d-g}. Then the dimension of

$G'\cap p^{-1}(C)$ is equal to $g+4(d-g-3)$ for $C\in p(G')$, which is the dimension of $G_0\cap p^{-1}(C)$ for $C\in p(G_0)$ $(m(G_0)=M_g)$. Since G' is a component of G_d^3, dim $G'\geq 3g-3+\rho(d,g,3)$ and hence dim $p(G')=3g-3$. Accordingly $G'=G_0$.

Consequently, for any $d>\frac{3}{4}g+3$ any other possible component of G other than G_0 arises from the family of the subseries g_d^3 of a special series $|D|$ only. Beause we consider only the case $\frac{3}{4}g+3<d\leq 2g-2$, $d\geq 7$ and if $d=7$, then $g=5$ and hence $r(D)\leq 3$ for any divisor of degree 7. As a result there is no component of $I'_{d,g,r}$ other than S_0 in case $d=7$. Therefore we may just assume that $d>7$. Here g_d^r is said to be special if $l(K\backslash D)>0$ for $D\in g_d^r$.

By the fact mentioned before, we will look for a component of G whose general series is a special series. For brevity, we call base point free simple series a birationally very ample series.

LEMMA 3. 1. *If an irreducible component W of G_d^2 contains a subvariety X with* dim $X>g-8$, *whose general member is birationally very ample, then the dimension of W is equal to the expected dimension* $3g-3+\rho(d,g,2)=3d+g-9$.

Proof. Let $\tilde{X}$ be the variety whose points are the pairs $(\mathcal{D},F)$, where $\mathcal{D}$ is a member of X, F is a frame for the corresponding $\mathcal{D}$. Then the tangent space $T_{(\mathcal{D},F)}(\tilde{X})$ to $\tilde{X}$ at $(\mathcal{D},F)$ is a vector subspace of $H^0(C,N_\phi)$ of dimension at least $(g-7)+3^2-1=g+1$, since $H^0(C,N_\phi)$ is the tangent space to $\tilde{W}$ at $(\mathcal{D},F)$ where $\tilde{W}$ is the variety associated with W.

Let Z be the ramification divisor of $\phi_{\mathcal{D}}$, for a general $\mathcal{D}\in X$. Then we have an exact sequence

$$0\rightarrow K_\phi\rightarrow N_\phi\rightarrow N_\phi'\rightarrow 0$$

where $K_\phi=\theta_C(Z)/\theta_C$, $N_\phi'=\phi^*\theta_{P^2}/\theta_C(Z)$. Then N_ϕ' becomes a line bundle, K_ϕ is concentrated on Z (See [2]). Accordingly $h^1(C,K_\phi)=0$ and so $h^1(C,N_\phi)=h^1(C,N_\phi')$. But by Lemma (1. 4) in [4],

$$T_{(\mathcal{D},F)}(\tilde{X})\cap H^0(C,K_\phi)=\{0\}.$$

Thus dim $T_{(\mathcal{D},F)}(\tilde{X})\leq h^0(C,N_\phi')$ and hence $r(N_\phi')\geq g$. If N_ϕ' is special, then by Clifford's theorem

$$\begin{aligned} h^1(C, N_\phi') &= r(N_\phi') + g - \deg(N_\phi') \\ &\leq g - r(N_\phi') \leq 0. \end{aligned}$$

Consequently N_ϕ' must be nonspecial, $h^1(C, N_\phi) = h^1(C, N_\phi') = 0$. Therefore W was the expected dimension $3g-3+\rho(d, g, 2) = 3d+g-9$, for $\dim T_{\mathscr{D}}(W) = 3g-3+\rho(d, g, 2) + h^1(C, N_\phi)$ (See [2]).

THEOREM 3.2. *Let $\mathscr{W}$ be an irreducible closed subvariety of G_d^r whose general series is a complete birationally very ample series for $r \geq 2$. Then*

$$\dim \mathscr{W} \leq 3d+g-4r-1$$

Proof. The above inequality holds for $r=2$. We may assume $r \geq 3$. Let C_{r-2} be the $(r-2)$-folds symmetric product over M_g of universal curve C. And let X be the set of the pairs $(\Sigma P_i, |D|)$ of birationally very ample $|D| \in \mathscr{W}$ and $\Sigma P_i \in C_{r-2}$ such that $|D| \backslash \Sigma P_i$ is birationally very ample g^2_{d-r+2}. Then X is open in $C_{r-2} \times_{M_g} \mathscr{W}$.

We define a morphism

$$\phi : X \to G^2_{d-r+2}$$

by $\phi(\Sigma P_i, |D|) = |D| \backslash \Sigma P_i$. Let us consider the fiber of a fixed $(\Sigma P_i$ $|D|)$. We suppose $\phi(\Sigma P_i, |D|) = \phi(\Sigma Q_i, |D'|)$, or equivalently, $|D| \backslash \Sigma P_i = |D'| \backslash \Sigma Q_i$. Then$||D| \backslash \Sigma P_i + \Sigma Q_i| = |D'|$ and hence $r(K \backslash D + \Sigma P_i \backslash \Sigma Q_i) = r(K \backslash D + \Sigma P_i)$. Thus ΣQ_i is a base locus of $|K \backslash D + \Sigma P_i|$ since $l(K \backslash D + \Sigma P_i) = l(K \backslash D) \neq 0$. There exists only the finite number of such ΣQ_i*'s*, accordingly such $|D'|$'s are finite, for $||D| \backslash \Sigma P_i + \Sigma Q_i| = |D'|$. Therefore the dimension of the fiber of ϕ is equal to zero. On the other hand, by Lemma 3.1.

$$\dim \phi(X) \leq 3(d-r+2) + g - 9.$$

Thus

$$\begin{aligned} \dim X &\leq 3d+g-3r-3 \\ \dim \mathscr{W} &\leq 3d+g-3r-3-r+2 \\ &= 3d+g-4r-1. \quad \square \end{aligned}$$

THEOREM 3.3. *Let $\mathscr{B}$ be the union of the components of G_d^3 whose general series is a birationally very ample series. Then $\mathscr{B}$ is irreducible* *and hence $\mathscr{B}=G=G_0$ for $d\geq g+3$. In particular G and $I'_{d,g,3}$ is irreducible.*

Proof. As mentioned before, any other component can come into being from only the family of the special series. Precisely suppose that $\mathscr{U}$ is another component of $\mathscr{B}$. Then a general series g_d^3 of $\mathscr{U}$ is a general subseries of a birationally very ample series $|D|$, where $|D|$ is a general member of some irreducible component $\mathscr{W}$ of G_d^r whose general member is birationally very ample complets series for some $r>d-g\geq 3$. Thus $\mathscr{U}$ is the family of the Grassmannians of projective 3-planes in the projective r-space $|D|$, $|D|\in\mathscr{W}$. Consequently

$$\begin{aligned}\dim \mathscr{U} &\leq 3d-4r+g-1+4(r-3)\\ &=3d+g-15\\ &<4d-15\\ &=3g-3+\rho(d,g,3)\end{aligned}$$

for $d\geq g+3$. This is a contradiction since every component of G_d^3 has dimension at least $4d-15$.

In order to deal with the case $d\leq g+2$, it is useful to have an upper bound of the dimension of $G_d^r(C)$ in a neighborhood of the birationally very ample series g_d^r on a fixed curve C. Fortunately we have the following effect in the proof of Proposition 2. 12 in [3].

THEOREM 3. 4. *Let C be a curve of genus g, D special divisor on C with $|D|$ a birationally very ample g_d^r. Then in a neighborhood of $|D|$ on the Jacobi variety $J(C)$ of C*

$$\dim W_d^r(C)\leq 2d-g-3r+l(K\backslash 2D)+1.$$

This bound is also given by J. Harris, P. Griffiths, R. Accola. (See [9])

REMARK. In case $d\leq g+2$, any component of $I'_{d,g,3}$ other than S_0 can be given by some component $\mathscr{W}$ of Theorem 3. 2 for $r>3$ or by a component V of G_d^3 with $p(V)\subsetneqq M_g$ whose general member is a complete g_d^3. Then for a general curve $C\in p(V)$, $V\cap p^{-1}(C)=G_d^3(C)$ has dimension greater than $\rho(d,g,3)$ by Theorem 2. 1. In other words, Ker $\mu_0\neq\{0\}$ for a general $g_d^3(=|D|)\in G_d^3(C)$ on a general curve $C\in p(V)$, since the dimension of the tangent space to $G_d^3(C)$ *at* $g_d^3(=|D|)$ is equal to $\rho(d,g,3)+\dim$ Ker μ_0, where

$$\mu_0: H^0(C,\mathscr{O}(D))\otimes H^0(C,K(\backslash D))\to H^0(C,K)$$

is the cup-product map. Therefore in case $d=g+2$, there is not such a V, for $l(K\backslash D)=h^0(C,K(\backslash D))=1$ implies Ker $\mu_0=\{0\}$.

LEMMA 3.5. *There is not a component of $G_{g+1}{}^3$ which does not dominate M_g and whose general member $g_{g+1}{}^3$ is complete.*

Proof. Assume that V is such a component of $G_{g+1}{}^3$. Let f be the degree of the base locus of $|K\backslash D|=|E|$ where $|D|$ is a general member of V. We denote by X the family of such a series $|D|$ of V.

If $f>0$, we have the following morphism

$$\psi : X \to G^1{}_{g-3-f}$$

defined by $\psi(g^3{}_{g+1}=|D|)=|K\backslash D|\backslash F$, where F is the base locus of $|K\backslash E|$. Thus

$$\begin{aligned}\dim V &\leq \dim G^1{}_{g-3-f}+f\\ &=3g-3+\rho(g-3-f,g,1)+f\\ &=4g-11-f\\ &\lneq 4(g+1)-15.\end{aligned}$$

This contradicts the fact that V is a component of $G^3{}_{g+1}$. Consequently $f=0$ and hence Ker $\mu_0=H^0(K\backslash 2E)$ where $|E|=|K\backslash D|$. A general curve C of $p(V)$ is non-hyperelliptic, for a hyperelliptic curve cannot have a special very ample series. Therefore dim Ker $\mu_0\leq 2$ by Clifford's theorem.

In dim Ker $\mu_0=h^0(C,K\backslash 2E)=2$, then we have the following morphism which is finite morphism onto its image

$$\psi : \to G_4{}^1$$

defined by $\psi(|D|)=|K\backslash 2E|$ where $E\in|K\backslash D|$. Thus

$$\begin{aligned}\dim V &\leq \dim G_4{}^1=2g+3\\ &\lneq 4(g+1)-15\end{aligned}$$

for $\rho(g+1,g,3)\geq 0$ implies $g\geq 8$. Accordingly this case cannot occur.

If dim Ker $\mu_0=1$, then $|K\backslash 2E|=P_1+P_2+P_3+P_4$. Thus for any divisor $E\in|K\backslash D|$ not containing any P_i, $i=1,\cdots,4$

$$\begin{aligned}\overline{\phi_4(E)} &\supset PT_{|E|}W_{g-3}{}^3(C)\\ &=PT_{|K\backslash 2E|}W_4{}^0(C)\\ &=\overline{\phi_K(P_1+P_2+P_3+P_4)}.\end{aligned}$$

Then $r(E+P_1+P_2+P_3+P_4)=r(E)+4$ by geometric Riemann-Roch theorem. This is impossible since $|D|$ is base point free. (In fact, our $|D|$ is very ample). Hence such a component V cannot exist in $G_{g+1}{}^3$. □

Now we will show that there is no component of $G_d{}^3$ whose general member $g_d{}^3$ is a subseries $g_d{}^3$ of a very ample ple series $g_d{}^r=|D|$ for some $r>3$. Then this implies the irreducibility of $I'_{d,q,3}$, as mentioned in the above Remark.

In the first place, it will be shown that there is no component of $I'_{g+2,g,3}$ which is given by the family of the series $(|D|,\tilde{C})$ with $l(K\backslash D)=2$. Recall that we denote by $\mathscr{B}$ the union of the components of $G_d{}^3$ whose general series is a birationally very ample series. We can give more general result as follows.

LEMMA 3.6. *In case $d=g+2$, there is no component in $\mathscr{B}$ whose general series is a subseries of some birationally very ample series $|D|=g_{g+2}{}^4$. In particular there is no component in G whose general member is a subseries of some very ample $|D|=g_{g+2}{}^4$.*

Proof. Assume that there is such a component $\mathscr{T}$ in $\mathscr{B}$. Then we can associate with $\mathscr{T}$ an irreducible component $\mathscr{W}$ of $G_{g+2}{}^4$ such that $g_{g+2}{}^3$ is a general member of $\mathscr{T}$ if and only if it is a general subseries $g_d{}^3$ of a general $|D|$ of $\mathscr{W}$. Let f be the degree of the base locus of $|K\backslash D|=|E|$, where $|D|$ is a general member of W. Then

$$|E|=|E\backslash F|+F(=g^1{}_{g-4-f}+F)$$

where F is the base locus of $|E|$. Let $\mathscr{W}'$ be the closure of the following set:

$\{|E|\in G^1{}_{g-4}|\,|E|=|K\backslash D|, |D|$ a general member of $\mathscr{W}\}$

$$\begin{aligned}\dim\mathscr{W}&=\dim \mathscr{W}'\\ &\leq\dim G^1{}_{g-4-f}+f\\ &=3g-3+\rho(g-4-f,g,1)+f\\ &=4g-13-f.\end{aligned}$$

Thus

$$\begin{aligned}\dim \mathscr{T}&=\dim \mathscr{W}+4(4-3)\\ &\leq 4g-13-f+4\\ &\lneq 4(g+2)-15.\end{aligned}$$

By Theorem 2.1 we obtain the desired result. □

Therefore we consider only the case of $l(K\backslash D)\geq 3$ for $d\geq g+2$.

THEOREM 3.7. *$\mathscr{B}$ is irreducible whenever $d\geq g+1$ and hence $\mathscr{B}=G=G_0$. In particular $I'_{d,g,3}$ is irreducible.*

Proof. By Therem 3.3, it is sufficient to consider the case $d=g+1$, $g+2$. Assume $\mathscr{U}$ is a component of $\mathscr{B}$ other than G_0. Then $\mathscr{U}$ comes from an irreducible closed subvariety $\mathscr{W}$ of $G_d{}^r$ such that $g_d{}^3$ is a general member of $\mathscr{U}$ if and only if it is a general subseries of a general $|D|$ of $\mathscr{W}$ for $r\geq d-g+3$ by Lemmas 3.5, 3.6. Let f be the degree of the base locus F of $|E|=|K\backslash D|$ where $(|D|, C)$ is a general member of $\mathscr{W}$.

We denote by $\mathscr{W}'$ the same object in the proof of Lemma 3.6. Then the following cases may occur.

Case 1) $|E\backslash F|=g^{g-d+r-1}_{2g-2-d-f}$ is birationally very ample. Then by Theorem 3.2

$$\begin{aligned}\dim \mathscr{W} &=\dim \mathscr{W}'\\ &\leq 3(2g-2-d-f)-4(g-d+r-1)+g-1+f\\ &=3g+d-2f-4r-3.\end{aligned}$$

As a result

$$\begin{aligned}\dim \mathscr{U} &\leq 3g+d-2f-4r-3+4(r-3)\\ &\lneq 4d-15\end{aligned}$$

for $d>g$. Thus this case cannot be occur.

Case 2) $|E|$ is not birationally very ample and the image curve C' of $\phi_{|E|}$ is not a rational curve. Let γ, n be the genus and the degree of C' respectively. Then the moduli dimension of curves, that is, dimension of $p(\mathscr{U})$ is no more than $2g+(2n-3)(1-\gamma)-2$ by Riemann's moduli count. Hence

$$\dim \mathscr{W}\leq (2g-2)+\dim W_d{}^r(C)$$

since $\gamma\geq 1$

$$\leq (2g-2)+2d-g-3r+1$$

by Theorem 3.4 for $d\geq g$. Thus

$$\begin{aligned}\dim \mathscr{U} &= \dim \mathscr{W} + 4(r-3)\\ &= 2d+g+r-13\\ &\lneq 4d-15\end{aligned}$$

since $r \leq \frac{2d-g+1}{3}$ by Castelnouvo's genus bound and $d>7$ as mentioned before. Consequently this case cannot occur either.

Case 3) $|E|$ is not birationally very ample and the genus of the image curve of $\phi_{|E|}$ is equal to zero.

Let n be the degree of the morphism $\phi_{|E|}$. Then $|E| = (i-1)g_n{}^1 + F$ and $f = (2g-2-d) - (i-1)n$, where $i = l(K\backslash D) = l(E) = g-d+r$. Therefore

$$\begin{aligned}\dim \mathscr{W} &\leq \dim G_n{}^1 + f\\ &\leq 2g-5+2n+2g-2-d-n(i-1)\\ &= 4g-d-n(i-3)-7\\ \dim \mathscr{U} &= \dim \mathscr{W} + 4(r-3)\\ &\leq 4g-d+n(i-3)-7+4(r-3)\\ &= 3d-(n-4)(i-3)-7\\ &\leq 3d+(g-d+r-3)-7\end{aligned}$$

since hyperelliptic curve has no special birationally very ample series and hence $n \geq 3$. Thus

$$\begin{aligned}\dim \mathscr{U} &\leq 2d+g+r-10\\ &\lneq 4d-15\end{aligned}$$

since $r \leq \frac{2d-g+1}{3}$, $d>8$.

As a result, in any case the existence of another component $\mathscr{U}$ leads to a contradiction to Theorem 2. 1.

4. Some examples

Let $M_{g,n}{}^1$ be the locus in M_g of curves having a base point free $g_n{}^1$, $n \leq \frac{g}{2}+1$. Then this family covers M_g. Then one may expect that the dimension of the component of $G_d{}^r(C)$ whose general member is birationally very ample with birationally very ample residual become decreasing with respect to n, where $C \in M_{g,n}{}^1$.

EXAMPLE 1. Let C be a curve which belongs to $M_{g,3}{}^1$. Then there is no birationally very ample series $|D|$ on C whose residual series $|K\backslash D|$ is also birationally very ample.

Proof. Suppose that both $|D|$ and $|K\backslash D|$ are birationally very ample series for some divisor D on C. Then two conditions are imposed on $|D|$ and $|K\backslash D|$ by any pair of two points except only the finite pairs.

Thus for general divisor $P+Q+R$ of $g_3{}^1$,

$$r(|D|\backslash P\backslash Q)=r(|D|\backslash P\backslash R)=r(D)-2$$
$$r(|K\backslash D|\backslash Q\backslash P)=r(|K\backslash D|\backslash Q\backslash R)=r(K\backslash D)-2.$$

Hence these are codimension one subspaces of $|D|\backslash P$ and $|K\backslash D|\backslash Q$ respectively. We choose D_0, E_0 in $|D|\backslash P$ and $|K\backslash D|\backslash Q$ such that D_0, E_0 does not belong to $\{|D|\backslash P\backslash Q\}\cup\{|D|\backslash P\backslash R\}$ and $\{|K\backslash D|\backslash Q\backslash P\}\cup\{|K\backslash D|\backslash Q\backslash R\}$ respectively. Then

$$(D_0, P+Q+R)=P, \quad (E_0, P+Q+R)=Q.$$

Consequently

$$(D_0+E_0, \ P+Q+R)=P+Q, \ D_0+E_0\in|K|.$$

This implies that $|P+Q+R|=g_3{}^1$ imposes 3 conditions on $|K|$, and hence contradicts Riemann-Roch Theorem.

On the other hand, it is useful to our problem to know a sharper upper bound of the dimension of $W_d{}^r(C)$ whose general $|D|$ is birationally very ample. But the following example satisfies the equality of the bound in Theorem 3.3.

EXAMPLE 2. Let C be a curve of genus≥ 11 with base point free $g_3{}^1$. Then for any $P\in C$, the residual series of $|g_3{}^1+P|=|F|$ becomes a birationally very ample series g^{g-4}_{2g-6}. In particular, in a neighborhood of the series

$$\dim \ W^{g-4}_{2g-6}(C)=1.$$

Proof. Severi gave the beautiful fact that if a curve of genus g is d_1, d_2 coverings of the genus g_1, g_2 curves respectively, then $g\leq(d_1-1)(d_2-1)+d_1g_1+d_2g_2$. By this C has a unique $g_3{}^1$ and can be neither hyperelliptic nor elliptic-hyperelliptic. And C cannot have a base point free g_5^2 nor a birationally very ample g_6^2, since base point free g_5^2 become birationally very ample and the genus of C is no less than 11. Conesequently $r(g_3^1+P)=r(F)$ is equal to one for any $P\in C$. And any g_6^2 is base point free, the degree of the morphism ϕ associated the series g_6^2 is equal to 3, for C is not elliptic-hyperelliptic. Then

$$|\phi^{-1}(P')|=g_3^1, \quad |\phi^{-1}(2P')|=|2g_3^1|=g_6^2, \quad P'\in\phi(C).$$

Accordingly any g_6^2 on C is equal to $|2g_3^1|$ and hence $r(g_3^1+P+Q+R)=r(F+Q+R)=1$ for any pair (Q, R) but one pair (Q_0, R_0) such that $P+Q_0+R_0=g_3^1$. Consequently the residual series $|K\backslash F|$ is birationally very ample because $r(K\backslash F\backslash Q\backslash R)=r(K\backslash F)-2$ except the pair (Q_0, R_0) by Riemann-Roch theorem. Therefore the dimension of the family of birationally very ample g^{g-4}_{2g-6} at least one and exactly one by Theorem 3. 4.

REMARK. Any g_6^2 on a trigonal curve C of genus$\geq$11, is equal to $|2g_3^1|$. Here we call curve $C\in M_{g,3}^1$ a trigonal curve.

We now give an interesting example such that $I'_{d,g,r}$ is reducible. Moreover this has other component whose dimension violates $\lambda(d, g, r)$ even in case $\rho(d, g, r)=g$.

LEMMA 4. *Let C be a trigonal curve of genus $\geq$ 11. $|K\backslash 2F|$ is very ample where $|F|=g_3^1$.*

Proof. Consider the dimension of $|2F+P+Q|$ for $P, Q\in C$. Then $r(2F+P+Q)\leq 3$ since any trigonal curve of genus$\geq$3 cannot be hyperelliptic. *If* $r(2F+P+Q)=3$, that is, $r(2F+P+Q)=g_8^3$, then this series cannot be birationally very ample by Castenouvo's genus bound. But if we assume that the degree of $\phi_{|2F+P+Q|}=2$, then the genus of the image curve is no more than one. This is impossible, for C cannot be hyperelliptic nor elliptic-hyperelliptic. Thus $r(2F+P+Q)=2$, equivalently, $|K\backslash 2F|$ is very ample since $r(K\backslash 2F\backslash P\backslash Q)=r(K\backslash 2F)-2$ for any (P, Q) by Riemann-Roch theorem.

EXAMPLE 5. For $d=2g-8$, $r=g-8$, $g\geq 18$, the only component of $I_{d',g,r}$ which is not the (unique) dominating component is the component given by the series whose residual is $g_6^2=|2g_3^1|$ on the locus of trigonal curves. Moreover its dimension violates the expected dimension $\lambda(2g-$

$8, g, g-8)$.

Proof. Let S be the component of $I'_{2g-8,g,g-8}$ which is not the dominating component. Then S is given by some component W of G_{2g-8}^{α} whose general member is complete very ample for $\alpha \geq g-7$. Then only the following cases may occur by Clifford's Theorem.

Case 1. $\alpha = g-7$

$$\dim S = \dim W + (g-6)(g-7) - 1$$

by Theorem 3.2

$$\begin{aligned} &\leq 3(2g-8) + g - 4(g-7) + (g-6)(g-7) - 2 \\ &\lneq \lambda(2g-8,\ g,\ g-8). \end{aligned}$$

Thus this case cannot occur.

Case 2. $\alpha = g-6$.

Let W' be the closure of the locus of the residual series $|K \backslash D|$ of a general series $|D|$ of W. Then general $|K \backslash D| = g_6^1$ in W' has the base locus of degree f, for some $f \geq 0$. Then

$$\begin{aligned} \dim W' &\leq \dim G_{6-f}^1 + f \\ &= 2g - 5 + 2(6-f) + f \\ &= 2g + 7 - f. \end{aligned}$$

Hence

$$\begin{aligned} \dim S &= \dim W + (g-5)(g-7) - 1 \\ &= \dim W' + (g-5)(g-7) - 1 \\ &\leq 2g + 7 + (g-5)(g-7) - 1 \\ &\lneq \lambda(2g-8,\ g,\ g-8). \end{aligned}$$

Consequently case 2 cannot occur.

Case 3. $\alpha = g-5$.

We denote by W' same one in case 2. Then a general $\mathcal{E} = g_6^2$ of W' is base point free and not birationally very ample, since $g(C) \geq 11$. If deg $\phi_{\mathcal{E}}$

$=2$, then the genus of the image curve C' of ϕ_g is no more than one, and hence the general curve C of S is elliptic-hyperelliptic. But in fact our curve C cannot be hyperelliptic, since hyperelliptic curve has no special very ample series. On the one hand, if C is elliptic-hyperelliptic, $g_6{}^2=\phi^*(g_3{}^2)$, where the linear series $g_3{}^2$ is given by the hyperplane sections of the genus 1 curve C'. Thus $|g_6{}^2+P+Q|=\phi^*(g_3{}^2+P')=\phi^*(g_4{}^3)=g_8{}^3$ for each pair (P, Q) such that $\phi(P)=\phi(Q)=P'$ for some $P'\in C'$. This is impossible since a general member of W is very ample. As a consequence, deg $\phi_g=3$. Therefore the image curve C' is a conic and hence C is a trigonal curve, $g_6{}^2$ is equal to $|2g_3{}^1|$ by the previous Remark. Thus a general series of W is residual to $|2g_3{}^1|$ on a trigonal curve. (Consequently there is no contradiction in the last case by Lemma 4.) As a result, there exists the component S whose general member is in fact a fiber of a trigonal curve. Thus

$$\begin{aligned}\dim S &= \dim M_{g,3}{}^1+0+(g-4)(g-7)-1\\ &=5g-20+(g-7)(g-7)-1\\ &\gneqq \lambda(2g-8, g, g-8)\end{aligned}$$

for $g\geq 18$. Consequently, we obtain the desired result.

References

1. Accola, R.D.M., *On Castelnouvo's inequality for algebraic curves. I*, Trans. Amer. Math. Soc. **251** (1979), 357-363.
2. Arbarello, E., Cornalba, M., *Su una congettra di Petri*, Comment. Math. Helvetici **56** (1981), 1-38.
3. Arbarello, E., Cornalba, M., *A few remarks*, Ann. Sci. Ec. Norm. Supper (4) **16** (1983), 467-488.
4. Arbarello, E., Cornalba, M., *Footnotes to a paper of Beniamino Segre*, Math. Ann. **256** (1981), 341-362.
5. Arbarello, E., Cornalba, M., Griffiths, P., Harris, J., *Geometry of algebraic curves*, Berlin-Heidelberg-New York-Tokyo, Springer 1984.
6. Eisenbud, D., Harris, J., *Divisors on a general curves and cuspidal rational curves*, Invent. Math. **74** (1983), 371-418.
7. Fulton, W., Lazarzfeld, R., *On the connectedness of degeneracy loci and special divisors*, Acta Math. **146** (1981), 271-283.
8. Griffiths, P., Harris, J., *On the variety of special linear systems on a general algebraic curves*, Duke Math. J. **47** (1980), 233-272.

9. Harris, J., Eisenbud, D., *Curves in projective space,* Les Presses de l'Université de Montréal, Montréal, 1982.

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