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SOME REMARKS ON CANONICAL TRANSFORMS REPRESENTING SL(2, R), IN $L^2(R^2)$

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1. Introduction

A series of articles[1-6] Programmed by M. Moshinsky, K.B. Wolf, and collaborators have shown that canonical transformations or symplectic transformations in quantum mechanics give a better understanding of dynamical groups for quantum systems together with unitary representations of SL$(2, \boldsymbol{R})$. In his book[7], K.B. Wolf treated in detail canonical transformations associated with SL$(2, \boldsymbol{R})$ in the form of integral transforms on $L^2(\boldsymbol{R})$. His method, due to M. Moshinsky and C. Quesne[1], is a quantization of the symplectic action of SL$(2, \boldsymbol{R})$ on $\boldsymbol{R}^2$, the phase space for a classical system of one degree of freedom. However, phase factors in integral transforms are determined only up to sign by the reason that the integral transforms give a unitary ray representation of SL$(2, \boldsymbol{R})$.

It was D. Shale[9] who observed that the quantization determines a double-valued unitary representation of the symplectic group Sp$(2n, \boldsymbol{R})$, which is analogous to the spin representation of the orthogonal group. This representation is nowadays known as the metaplectic representation, or as the Segal-Shale-Weil representation[11]. In view of this, the integral transforms on $L^2(\boldsymbol{R})$ associated with a symplectic group SL$(2, \boldsymbol{R})$ should be a unitary representation of the double covering SL$_2(2, \boldsymbol{R})$ of SL$(2, \boldsymbol{R})$. It is to be noted here that while SL$(2, \boldsymbol{R})$ is a "spin" group for SO$_0(1, 2)$, one should consider a double covering (or "spin" group) of SL$(2, \boldsymbol{R})$ still more.

The integral transforms for Sp$(2n, \boldsymbol{R})$ are determined in Ref. 12 up to sign. The metaplectic representation of the inhomogeneous symplectic group was investigated in Ref. 14, and for some one-parameter subgroups,

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phase factors in integral transforms were determined by computing explicitly the Maslov indices[11]. To our knowledge, the phase factor in the integral transform representing $SL_2(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R})$ seems not to have been explicitly expressed yet, nor the composition of the integral transforms does.

Roughly speaking, our method is as follows. The two-fold covering causes a sign problem in the integral kernel. Though the problem has been solved in an abstract manner, it seems to require further investigation in an explicit manner. Canonical transforms representing unitarily $SL_2(2, \boldsymbol{R})$, constructed in the form of integral transforms in $L^2(\boldsymbol{R})$ on the composition of canonical transforms. From the viewpoint of reducing a quantum system, the quantization in $L^2(\boldsymbol{R}^2)$ will also provide us with a unitary representation of $SL(2, \boldsymbol{R})$ which is a direct sum of unitary irreducible representations in the space of functions of radical variable. In the course of discussion it will be found why $SL(2, \boldsymbol{R})$ is represented, and no covering groups appear.

2. Oscillator Realization

We start with a classical dynamical system of one degree of freedom. Let $(\boldsymbol{R}^2, \omega)$ be a standard symplectic vector space, where ω is given in the Cartesian coordinates (x, p) of $\boldsymbol{R}^2$ by $\omega = dp \wedge dx$. The linear group of the symplectic transformations of $(\boldsymbol{R}^2, \omega)$ is isomorphic to $SL(2, \boldsymbol{R})$. Let the Lie algebra $sl(2, \boldsymbol{R})$ of $SL(2, \boldsymbol{R})$ be written as

$$(2.1) \qquad \xi = 1/2\begin{bmatrix} c_2 & c_1+c_3 \\ c_1-c_3 & -c_2 \end{bmatrix}, \quad c_j \in R$$

We take a basis $\{e_j\}$, $j=1, 2, 3$, of $sl(2, \boldsymbol{R})$ so that any element ξ given by (2.1) can assume the form $\xi = \Sigma c_j e_j$.

Let ξ be given by (2.1). The one-parameter subgroup $\exp(t\xi)$ of $SL(2, R)$ induces an infinitesimal symplectic transformation $\xi_Q(\omega)$ at $\omega \in Q = \boldsymbol{R}^2$ by

$$(2.2) \qquad \xi_Q(\omega) = \frac{d}{dx}\exp(t\xi)\cdot\omega|_{t=0},$$

which takes the form

$$(2.3) \quad \xi_Q(\omega) = 1/2\{c_2x + (c_1+c_3)p\}\frac{\partial}{\partial x} + 1/2\{(c_1-c_3)x - c_2p\}\frac{\partial}{\partial p}.$$

The ξ_Q is a Hamiltonian vector field having a Hamiltonian or generating function f determined by the condition $i(\xi_Q)\omega=-df$, $i(\cdot)$ denoting the interior product. In fact, the equations, equivalent to $i(\xi_Q)\omega=-df$,

$$\frac{\partial f}{\partial p}=1/2\{c_2x+(c_1+c_3)p\},\quad \frac{\partial f}{\partial x}=-1/2\{(c_1-c_3)x-c_2p\} \tag{2.4}$$

are easily integrated to give

$$f=c_1f_1+c_2f_2+c_3f_3$$

with

$$f_1=1/4(p^2-x^2),\quad f_2=1/4(xp+px),\quad f_3=1/4(p^2+x^2). \tag{2.5}$$

The functions f_j satisfy the sl(2, $\boldsymbol{R}$) commutation relations under the Poisson bracket,

$$\{f_1, f_2\}=-f_3,\quad \{f_2, f_3\}=f_1,\quad \{f_3, f_1\}=f_2. \tag{2.6}$$

We make some remarks on f_k, s. The f_3, half the Hamiltonian for the harmonic oscillator, is a Hamiltonian function for the Hamiltonian vector field induced by the action of the compact subgroup SO(2);

$$\exp(te_3)=\begin{bmatrix}\cos(t/2) & \sin(t/2)\\ -\sin(t/2) & \cos(t/2)\end{bmatrix} \tag{2.7}$$

The function f_1+f_3 is the Hamiltonian of a free particle which is related with exp $t(e_1+e_3)$.

We now turn to quantum mechanics. Following the Schrodinger procedure ($p\to -id/dx$), we obtain for the classical observables f_k the quantized operators J_k, $k=1, 2, 3$, fulfilling the sl(2, $\boldsymbol{R}$) commutation relations;

$$J_1=1/4\{-d^2/dx^2-x^2\},\quad J_2=1/(2i)(xd/dx+1/2),\quad J_3=1/4\{-d^2/dx^2+cx^2\}, \tag{2.8}$$

$$[J_1, J_2]=-iJ_3,\quad [J_2, J_3]=iJ_1,\quad [J_3, J_1]=iJ_2. \tag{2.9}$$

We define the operators J_+ and J_- as usual by

(2.10) $$J_+=J_1+iJ_2,\quad J_-=J_1-iJ_2,$$

respectively. On introducing oscillator annihilation and creation operators by

(2.11) $$a=-i/(\sqrt{2})(x+d/dx),\quad a^+=i/(\sqrt{2})(x-d/dx),$$

the operators J_+, J_-, and J_3 takes the form

(2.12) $$J_+=1/2(a^+)^2,\quad J_-=1/2(\mathrm{a})^2,\quad J_3=1/2(a^+a+1/2).$$

We are interested in whether SL$(2, \boldsymbol{R})$ will come out or not when the Lie algebra sl$(2, \boldsymbol{R})$ spanned by $J_k, k=1, 2, 3$, is exponentiated, that is, whether SL$(2, \boldsymbol{R})$ is realized in $L^2(\boldsymbol{R})$ as unitary operators or not. Generally speaking, a covering group of SL$(2, \boldsymbol{R})$ will come out. Incidentally, since the group manifold SL$(2, \boldsymbol{R})$ is homeomorphic to the product SO$(2)\times\boldsymbol{R}^2$, we may concentrate our attention to the subgroup SO(2) as far as the homotopy is concerned. Thus we pick up the operator J_3 associated with $\exp(te_3)$. Let $|j\rangle$, $j=0, 1, 2, \cdots$, be normalized eigenfunctions for the harmonic oscillator, which constitute a basis of $L^2(\boldsymbol{R})$. The $|j\rangle$'s are generated from the vacuum state $|0\rangle$;

(2.13) $$|j\rangle=(j!)^{-1/2}(a^+)^j|0\rangle,$$
$$J_3|j=1/2(j+1/2)|j\rangle.$$

As is well known, the operator J_3 has a self-adjoint extension and generate a unitary operator e^{-itJ_3}. From (2.13) we have

(2.14) $$e^{-itJ_3}|j\rangle=e^{-1/2(j+1/2)t}|j\rangle.$$

Since the right-hand side of Eq. (2.14) is a periodic function of t with period 8π, so is the left-hand side. That is, e^{-itJ_3} is periodic in t with period 8π. This is in marked contrast with the fact that the classical correspondent exp te_3 is periodic in t with period 4π. Thus we may understand that e^{-itJ_3} is a double covering of exp te_3 as one-parameter groups. The double periodicity was already pointed out in Ref. 14. Accordingly, we infer that the quantization by the Schrodinger procedure will lead to a representation of the two fold covering SL$_2(2, \boldsymbol{R})$ of SL$(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R})$.

Two fold covering can be observed from an article of V. Bargmann [15] on the representation of SL(2, $\boldsymbol{R}$), according to which the eigenvalues of J_3 would be half integers, if the quantized operators J_k gave an infinitesimal representation of SL(2, $\boldsymbol{R}$). However, Eq. (2.13) shows that J_3 has quarter integers as eigenvalues. This means that two fold covering of SL(2, $\boldsymbol{R}$) appears. For the sake of comparison we here remark that one obtains representations of $SO_0(1, 2)$ in the case of integer eigenvalues and SL(2, $\boldsymbol{R}$), two fold covering of $SO_0(1, 2)$, in the case of pure half integer eigenvalues.

We turn to the operators J_+ and J_- in order to gain an insight into the realization of sl(2, $\boldsymbol{R}$) by the Schrodinger quantization procedure. We have to point out here that V. Bargmann's theory [15] of the representation of SL(2, $\boldsymbol{R}$) is applicable to the representation of $SL_2(2, \boldsymbol{R})$ with a slight modification that J_3 has quarter integer eigenvalues. The modified Bargmann's theory will be applied in the sequel. Now we operate $|j\rangle$ with J_+ and J_- to get

(2.15) $$J_+|j\rangle=\{(j+2)(j+1)/4\}^{1/2}|j+2\rangle,$$
$$J_-|j\rangle=\{j(j-1)/4\}^{1/2}|j-2\rangle.$$

Equations (2.15) imply that $J_-|j\rangle=0$ for $j=0, 1$, so that our realization of sl(2, $\boldsymbol{R}$) is of positive type discrete class according to Bargmann's classification. Furthermore, Eq. (2.15) imply that $L^2(\boldsymbol{R})$ becomes a direct sum of two closed invariant subspaces spanned by $|j\rangle$ with j even and by $|j\rangle$ with j odd. As is easily seen from (2.13), the former is the space of even functions, and the latter the space of odd functions.

We proceed to the Casimir operator defined by

(2.16) $$\begin{aligned} Q&=(J_1)^2+(J_2)^2-(J_3)^2 \\ &=1/2(J_+J_-+J_-J_+)-(J_3)^2. \end{aligned}$$

Calculation in terms of the annihilation and creation operators results in $Q=3/16$, that is, Q reduces to a number. According to V. Bargmann [15] the eigenvalues of Q can be expressed as $q=n(1-n)$ with n quarter integers for the irreducible unitary representations of discrete class. We have here used the modified theory for $SL_2(2, \boldsymbol{R})$. In our case, we obtain $n(1-n)=3/16$, which is satisfied by the numbers $n=1/4$ and $3/4$. These numbers indeed characterize the realization of $SL_2(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R})$.

Now we denote the eigenvalues of J_3 by m. Then from Eq. (2.13) we have

$$(2.17)\qquad m=j/2+1/4 \text{ or } m=(j-1)/2+3/4.$$

Let $|j\rangle=|m;n\rangle$, where $n=1/4$ or $3/4$ according as j is even or odd. Then for both $n=1/4$ and $3/4$, Eqs. (2. 13) and (2. 15) turn out to be

$$(2.18)\qquad \begin{aligned} J_+|m;n\rangle &= \{3/16+m(m+1)\}^{1/2}|m+1;n\rangle, \\ J_-|m;n\rangle &= \{3/16+m(m-1)\}^{1/2}|m-1;n\rangle, \\ J_3|m;n\rangle &= m\ \ m;n\rangle. \end{aligned}$$

These equations provide irreducible infinitesimal unitary representations of $SL_2(2, \boldsymbol{R})$ of positive type discrete class $D_{1/4}{}^+$ and $D_{3/4}{}^+$. Hence the Schrodinger quantization procedure gives rise to realization of $SL(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R})$ which is a direct sum of $D_{1/4}{}^+$ and $D_{3/4}{}^+$. We point out in conclusion that $D_n{}^+$, $n=1/4$, $3/4$, were realized in Bargmann's Hilbert space of analytic functions by C. Itzykson [16].

3. Canonical Transforms

In this section we review quantum mechanical canonical transformations associated with $SL(2, \boldsymbol{R})$. Let us regard x and p as operators ($p=-id/dx$). We assume that a unitary operator U induces a linear transformation of x and p;

$$(3.1)\qquad U\begin{bmatrix} x \\ p \end{bmatrix}U^{-1}=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix},\quad ad-bc=1.$$

The transformation (3. 1) corresponds to the canonical transformation on $\boldsymbol{R}^2$ by $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the classical system. The unitary operator U satisfying (3. 1) is a canonical transformation for the quantum system in the sense that the canonical commutation relations between x and p are unaltered under U. We assume further that U is expressed as an integral transform; for $f\in L^2(\boldsymbol{R})$

$$(3.2)\qquad Uf(x)=\int_{\boldsymbol{R}} K(x, y)f(y)dy.$$

The identities $UxU^{-1}Uf=Uxf$ and $UpU^{-1}Uf=Upf$ for functions in a suitable domain yield sufficient conditions for the kernel to satisfy;

$$(3.3)\qquad \left(dx-b/i\frac{\partial}{\partial x}\right)K(x,y)=yK(x,y),$$
$$\left(-cx+a/i\frac{\partial}{\partial x}\right)K(x,y)=-1/i\frac{\partial}{\partial y}K(x,y).$$

Proposing a solution of the form $\exp[Ax^2+2Bxy+Cy^2]$, we find that Eqs. (3. 3) are satisfied by $A=id/(2b)$, $B=-i/b$, and $C=ia/(2b)$ for $b\neq 0$. The complex constant γ is determined up to a phase factor by the unitary condition

$$(3.4)\qquad \int_R \overline{K(x,z)}K(x,y)dx=\delta(z-y)$$

to be $\gamma=(2|b|)^{-1/2}$, where the bar over $K(x,z)$ indicates thee complex conjugate. Thus we have a solution for $b\neq 0$

$$(3.5)\quad K(x,y)=\alpha\{(2\pi|b|)^{-1/2}\}\exp[i(2b)^{-1}(dx^2-2xy+ay^2)]$$

where α is a complex constant with $|\alpha|=1$. To determine α is our purpose in the succeeding sections. The integral transform with the kernel (3. 5), though α being undetermined, allows for the inversion formula for rapidly decreasing functions, which can be proved by the Fourier integral theorem. Therefore, the integral transform becomes an isometry in a dense domain in $L^2(\boldsymbol{R})$, and extensible to the whole space $L^2(\boldsymbol{R})$ as a unitary operator.

On supposing that α is determined in some way or other, we calculate compositions of the integral transforms. Let

$$(3.6)\qquad M_{21}=M_2M_1,\quad M_k=\begin{bmatrix} a_k & b_k \\ c_k & d_k\end{bmatrix},\quad k=1,2,21,$$

By U_k, $k=1,2,21$, we mean the unitary operator determined by (3. 1) with coefficient matrices M_k^{-1}. For the unitary operators U_k, the integral kernels and the phase factors are denoted by K_k and α_k, respectively, $k=1,2,21$. We write out the iterated integral to get

$$(3.7)\qquad U_2U_1f(x)=\int dyK_2(x,y)\int K_1(y,z)f(z)dz.$$

For functions of compact support we can reverse the order of integration to integrate $K_2(x,y)K_1(y,z)$ with respect to y by using a formula

$$(3.8)\qquad \int_{-\infty}^{\infty} e^{1Ay}dy=\begin{bmatrix}(\pi/|A|)^{1/2}e^{i\pi/4} \text{ for } A>0,\\ (\pi/|A|)^{1/2}e^{-i\pi/4} \text{ for } A<0.\end{bmatrix}$$

From (3.7) and (3.8) it follows that

$$(3.9)\qquad K_2(x,y)K_1(y,z)dy = \begin{cases} \alpha_1\alpha_2(\alpha_{21})^{-1}e^{i\pi/4}K_{21}(x,z) & \text{for } b_{21}/(b_1b_2)>0, \\ \alpha_1\alpha_2(\alpha_{21})^{-1}e^{-i\pi/4}K_{21}(x,z) & \text{for } b_{21}/(b_1b_2)<0. \end{cases}$$

Since the functions of compact support are dense in $L^2(\boldsymbol{R})$, Eqs. (3.7) and (3.9) imply that

$$(3.10)\qquad U_2U_1 = \begin{cases} \alpha_1\alpha_2(\alpha_{21})^{-1}e^{i\pi/4}U_{21} & \text{for } b_{21}/(b_1b_2)>0, \\ \alpha_1\alpha_2(\alpha_{21})^{-1}e^{-i\pi/4}U_{21} & \text{for } b_{21}/(b_1b_2)<0. \end{cases}$$

Our purpose in this article is to show that for suitably chosen phase factors α_k, Eq. (3.10) exactly becomes $U_2U_1=U_{21}$. Then we will obtain not a unitary ray representation but a unitary representation of $SL_2(2,\boldsymbol{R})$.

4. Harmonic Oscillator

As we pointed out in Sec. 2, the compact subgroup $SO(2)$ is crucial in studying the realization of the double covering $SL_2(2,\boldsymbol{R})$ of $SL(2,\boldsymbol{R})$ in $L^2(\boldsymbol{R})$. Suppose that U_t is a unitary operator associated with the one-parameter group $\exp(2te_3)$. Then from (3.5) the integral transform U_t has the kernel

$$(4.1)\qquad K_t(x,y)=\alpha(t)(2\pi|\sin t|)^{-1/2} \cdot\exp[i(2\sin t)^{-1}(x^2+y^2\cos t-2xy)].$$

The phase factor $\alpha(t)$ can be determined under the condition that U_t should be a one-parameter group of unitary operators with initial value $U_0=id$. We notice that $\alpha(t)$ must be determined in each interval $n\pi<t<(n+1)\pi$, $n=0, \pm1, \pm2, \cdots$. We first determine $\alpha(t)$ in the intervals $-\pi<t<0$ and $0<t<\pi$ so that U_t can tend to the identity as t tends to zero, that is,

$$(4.2)\qquad K_t(x,y)\to\delta(x-y) \text{ as } t\to0.$$

To this end, we have only to require that $\int K_t(x,y)dy$ tends to one as t tends to zero. Calculation with the help of (3.8) results in

$$(4.3)\qquad \alpha(t)=\begin{cases} e^{i\pi/4} & \text{for } -\pi<t<0, \\ e^{-i\pi/4} & \text{for } 0<t<\pi. \end{cases}$$

We proceed to extend the domain of $\alpha(t)$ so that U_t can be continuous in t. First we look into the limit of $K_t(x,y)$ as t tends to π from the below. After the change of variables from t to s by $t=s+\pi$ the kernel reads

$$(4.4)\qquad K_{s+\pi}(x,y)=e^{-i\pi/4}(2\pi|\sin s|)^{-1/2}\exp[i(2\sin s)^{-1}(x^2+y^2)\cos s+2xy)].$$

Letting $s\to-0$, we find that

$$(4.5)\qquad K_{s+\pi}(x,y)\to-i\delta(x+y).$$

In order to define $\alpha(t)$ for $\pi<t<2\pi$, we have to impose the condition that

$$(4.6)\qquad K_t(x,y)\to-i\delta(x+y)\ \text{ as } t\to\pi+0.$$

By using the formula (3.8) we obtain eventually

$$(4.7)\qquad \alpha(t)=-ie^{-i\pi/4}\ \text{ for } \pi<t<2\pi.$$

In the same manner as above, we can determine $\alpha(t)$ for all the intervals $n\pi<t<(n+1)\pi$, $n=0,\pm1,\pm2,\cdots$, and the limits of $K_t(x,y)$ as $t\to n\pi$;

$$\alpha(t)=e^{-i\pi/4}\beta(t),\quad \alpha(t+4n\pi)=\alpha(t),$$

with

$$\beta(t)\to\begin{cases}-1 & \text{for } -2\pi<t<-\pi,\\ i & \text{for } -\pi<t<0,\\ 1 & \text{for } 0<t<\pi,\\ -i & \text{for } \pi<t<2\pi,\end{cases}$$

and

$$K_t(x,y)\to\begin{cases}-\delta(x-y) & \text{as } t\to-2\pi,\\ i\delta(x+y) & \text{as } t\to-\pi,\\ \delta(x-y) & \text{as } t\to0,\\ -i\delta(x+y) & \text{as } t\to\pi.\end{cases}$$

Thus we have defined U_t completely. In the course of determining $\alpha(t)$, we have found that U_t is periodic in t with period 4π.

The next task we have to fulfill is to show that the U_t's form a one-parameter group; $U_{s+t}=U_sU_t$. To this end, we emply the formula (3.10). For the integral transformations U_t under investigation, we have

$$U_sU_t=\alpha(s)\cdot\alpha(t)\cdot\alpha(s+t)\exp(\pm i\pi/4)U_{s+t}, \tag{4.10}$$

where the signs $\pm$ depend on the sign of $b_{21}/(b_1b_2)$ with $b_1=\sin t$, $b_2=\sin s$, and $b_{21}=\sin(s+t)$. To identify the sign of $b_{21}/(b_1b_2)$, we draw in the (s, t)-plane lines $s=n\pi$, $t=m\pi$, and $s+t=k\pi$, where n, m, and k are integers. In each triangular region enclosed by the lines drawn above, the sign of $b_{21}/(b_1b_2)$ is determined definitely. We can then conclude from (3.10), (4.8), and (4.10) that

$$U_sU_t=U_{s+t}. \tag{4.11}$$

On account of (4.9) this conclusion is true in the case where s and/or t are equal to $n\pi$.

Thus we know that U_t's form a one-parameter group of unitary operators. We show that the infinitesimal generator of U_t is indeed i times the Hamiltonian operator H for the harmonic oscillator, or its self-adjoint extension precisely. In effect, we can verify, after calculation, that the kernel K_t of theintegral transform U_t satisfies

$$\frac{\partial}{\partial x}K_t(x, y)=-iHK_t(x, y), \tag{4.12}$$

so that $U_t=e^{-itH}$. Further, in a suitable domain we obtain from (3.1) with $M=\exp(2te_3)$

$$e^{-itH}\begin{bmatrix} x \\ p \end{bmatrix}e^{itH}=\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \tag{4.13}$$

which is Heisenberg's picture of the harmonic oscillator. We remark incidentally that, while e^{-itH} is periodic in t with period 4π, the left-hand side of Eq. (4.13) is periodic in t with period 2π, so is the right-hand side.

The integral transform for the harmonic oscillator is found in Ref. 17, for example, in which the amplitude of K_t was given by $(2\pi|\sin t|)^{-1/2}$ but its double-valuedness was not clear. Then double-valuedness has been solved in Ref. 18 for the n-dimensional harmonic oscillator by employing the Maslov index. We have discussed the unitary operator e^{-itH} rather lengthly for the reason that the discussion in this section gives a key principle in studying the canonical transforms associated with SL$(2, \boldsymbol{R})$.

5. Dilation and a Free Particle

SL$(2, \boldsymbol{R})$ admits an Iwasawa decomposition, SL$(2, \boldsymbol{R})\cong$KAN, or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\tau}\cos\theta & e^{\tau}\zeta\cos\theta + e^{-\tau}\sin\theta \\ -e^{\tau}\sin\theta & -e^{\tau}\zeta\sin\theta + e^{-\tau}\cos\theta \end{bmatrix} \tag{5.1}$$

The parameters (θ, τ, ζ) are given by

$$e^{i\theta} = (a-ic)(a^2+c^2)^{-1/2}, \quad e^{\tau} = (a^2+c^2)^{1/2}, \quad \zeta = (ab+cd)(a^2+c^2)^{-1}. \tag{5.2}$$

We have done with the compact subgroup K in Sec. 4. We are going to give one-parameter groups of unitary operators associated with the subgroups A and N, which are parameterized by τ and ζ, respectively. Since the parameters τ and ζ are coordinates of the factor space $\boldsymbol{R}^2$ in the topological decomposition $SO(2)\times\boldsymbol{R}^2$ of $SL(2, \boldsymbol{R})$, we will not encounter the problem of covering.

We start with the subgroup A. For A, the unitary operators not expressed as the integral transforms given in (3.5), because $b=0$. We note here that group A is written as $\exp(2\tau e_2)$, so that the quantized operator $D=2J_2$ associated with $\exp(2\tau e_2)$ is expected to generate a one-parameter group $e^{-i\tau D}$, which describes a geometric transformation $x \to e^{\tau}x$. We can easily verify that the expression

$$e^{-i\tau D}f(x) = e^{-\tau/2}f(e^{-\tau}x) \tag{5.3}$$

defines a one-parameter group of unitary operators $e^{-i\tau D}$. In fact, the mappings $f(x) \to e^{-\tau/2}f(e^{-\tau}x)$ are unitary and form a one-parameter group. Moreover, one has for smooth functions $f(x)$

$$\frac{\partial}{\partial t}e^{-i\tau D}f(x) = -iDe^{-i\tau D}f(x). \tag{5.4}$$

and in a suitable domain

$$e^{-i\tau D}\begin{bmatrix} x \\ p \end{bmatrix} e^{i\tau D} = \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}. \tag{5.5}$$

We note that if we want to express $e^{-i\tau D}$ as an integral transform, we have the kernel $K_\tau(x, y)$ written as

$$K_\tau(x, y) = e^{-i\pi/2}\delta(e^{-\tau}x - y). \tag{5.6}$$

The $e^{-i\tau D}$ can be considered as describing a quantization of dilation in $\boldsymbol{R}$.

We turn to the subgroup N, which is written in the form $\exp \zeta(e_1+e_3)$. From (3.5) we have the integral kernel

$$K_\zeta(x,y)=\alpha(\zeta)(2\pi|\zeta|)^{-1/2}\exp[i(2\zeta)^{-1}(x-y)^2]. \tag{5.7}$$

The constant $\alpha(\zeta)$ is determined by the condition that $K_\zeta(x,y)$ tends to $\delta(x-y)$ as $\zeta\to 0$. As is Sec. 4, the formula (3.8) then gives

$$\alpha(\zeta)=e^{-i\pi(\mathrm{sgn}\zeta)/4}. \tag{5.8}$$

We can verify that the integral transforms with kernel $K_\zeta(x,y)$ form a one-parameter group by employing the formula (3.10) with $b_1=\zeta_1$, $b_2=\zeta_2$, and $b_{21}=\zeta_2+\zeta_1$. The infinitesimal generator of the one-parameter group is known to be $-i$ times the free particle Hamiltonian $F=J_1+J_3$, since

$$K_\zeta(x,y)=i/2\frac{\partial^2}{\partial x^2}K_\zeta(x,y). \tag{5.9}$$

Thus we have a one-parameter group of unitary operators $e^{-i\zeta F}$, which describes the time evolution for a free particle.

6. Canonical Transforms Representing $SL_2(2,\boldsymbol{R})$

In previous sections we have studied the one-parameter groups of unitary operators corresponding to all the subgroups in the Iwasawa decomposition (5.1) of $SL(2,\boldsymbol{R})$. That is, we have $e^{-i\theta H}$, $e^{-i\tau D}$, and $e^{-i\zeta F}$ for $\exp(2\theta e_3)$, $\exp(2\tau e_2)$, and $\exp\zeta(e_1+e_3)$, respectively. Hence, the composition $e^{-i\zeta F}e^{-i\tau D}e^{-i\theta H}$ will give a unitary representation of $SL_2(2,\boldsymbol{R})$, the double covering of $SL(2,\boldsymbol{R})$. We notice again that $SL_2(2,\boldsymbol{R})$ is represented, because the one-parameter group $e^{-i\theta H}$ requires the range of θ to be the double of the range of θ for $\exp(2e_3)$, the other the same ranges, $-\infty<\tau,\zeta<+\infty$. In view of this, we take (θ,τ,ζ) to be the parameters of $SL_2(2,\boldsymbol{R})$ with ranging over the interval of length 4π.

Indeed of composing the one-parameter groups, we are going to determine the factor α in the expression (3.5) for the parameters (θ,τ,ζ) of $SL_2(2,\boldsymbol{R})$. Let

$$\alpha=e^{-i\pi/4}\beta. \tag{6.1}$$

For $e^{-i\theta H}$, the values of β are ± 1, $\pm i$, and for $e^{-i\zeta F}$, $\beta=1$ or i, depending on the sign of b. In view of this, α is expected to range over

± 1, and $\pm i$.

Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in SL(2, $\boldsymbol{R}$), which is assigned by the parameters (5. 1). For M, there are two elements in $SL_2(2, \boldsymbol{R})$ which are assigned by the parameters (θ, τ, ζ) and $(\theta \pm 2\pi, \tau, \zeta)$, the sign $\pm$ depending on the value of β. To determine β, we consider the sign of

$$(6.2) \qquad b=\zeta e^{\tau} \cos\theta + e^{-\tau} \sin\theta$$

for τ and ζ fixed arbitrarily. On setting

$$(6.3) \qquad \tan\chi_0 = -\zeta e^{2\tau} \text{ with } |\chi_0| < \pi/2,$$

the sign of b is alternating, as θ goes into intervals $\chi_0 + n\pi < \theta < \chi_0 + (n+1)\pi$ in a manner such that, for example, $b>0$ for $\chi_0 < \theta < \chi_0 + \pi$. In view of the values of β for $e^{-i\theta H}$, we define β to take the values such that

$$(6.4) \qquad \beta \to \begin{cases} -1 & \text{for } \chi_0 - 2\pi < \theta < \chi_0 - \pi, \\ i & \text{for } \chi_0 - \pi < \theta < \chi_0 \\ 1 & \text{for } \chi_0 < \theta < \chi_0 + \pi, \\ -i & \text{for } \chi_0 + \pi < \theta < \chi_0 + 2\pi. \end{cases}$$

If $\tau = \zeta = 0$, then $\chi_0 = 0$, so that Def. (6. 4) reduces to (4. 8). We turn to the case of $\theta = \tau = 0$, whence $\tan\chi_0 = -\zeta$ with $|\chi_0| < \pi/2$. If $\zeta > 0$, then $\chi_0 < 0$, so that $\chi_0 < 0 < \chi_0 + \pi$. Hence, from (6. 4) we have $\beta = 1$. If $\zeta < 0$, then $\chi_0 > 0$, so that $\chi_0 - \pi < 0 < \chi_0$. Hence, $\beta = i$. Thus, Def. (6. 4) covers the values of β for $e^{-i\zeta F}$. We make an additional remark on β. The β takes one of values ± 1 (resp. $\pm i$) when $b>0$ (resp. $b<0$). Which of ± 1 (or $\pm i$) for β to take would be undetermined, if θ ranged over an interval of length 2π. We point out here that in Ref. 7 the amplitude $\alpha/(2\pi|b|)^{1/2}$ in the integral kernel (3. 5) is expressed as $e^{-i\pi/4}/(2\pi|b|)^{1/2}$. Our definition (6. 4) of resolves the ambiguity in the square root.

We wish to show that the integral transforms whose kernels (3. 5) has the factor $\alpha = e^{-i\pi/4}$ with β defined by (6. 4) indeed give a unitary representation of $SL_2(2, \boldsymbol{R})$. Let M_k, $k=1, 2, 21$, be matrices in SL(2, $\boldsymbol{R}$) with parameters $(\theta_k, \tau_k, \zeta_k)$ such that $M_{21} = M_2 M_1$. Then from (5. 1) and (5. 2) applied for $M_{21} = M_2 M_1$, we have

$$(6.5) \qquad \theta_{21} = \theta_2 + \arg[e^{\tau_1+\tau_2}(\cos\theta_1 - \zeta_2 \sin\theta_1) + ie^{\tau_1-\tau_2} \sin\theta_1].$$

When the argument and the equality in Eq. (6. 5) are taken modulo 2π, Eq. (6. 5) gives the parameter θ_{21} for M_{21}. However, if we take (6. 5) modulo 4π, the θ_{21} should be considered as a parameter of $SL_2(2, \boldsymbol{R})$. Moreover, Eq. (6. 5) modulo 4π gives the multiplication law for $SL_2(2, \boldsymbol{R})$ together with similar equations obtained from the second and third equations in Eq. (5. 2).

We are now looking into Eq. (6. 5). Let $Z(\theta_1)$ denote the quantity in (6. 5) enclosed by the square brackets, values of τ_1, τ_2, and ζ_2 being fixed. Let $\cot\kappa_1=\zeta_2$ with $|\kappa_1|<\pi/2$ for ζ_2 given. Then for $\zeta_2>0$, $\arg Z(\theta_1)$ ranges from -2π to 2π, when θ_1 goes from -2π to 2π, with $\arg Z(\kappa_1+n\pi)=n\pi+\pi/2$ and $\arg Z(n\pi)=n\pi$, $n=-2, -1, 0, 1$. For $\zeta_2<0$, one has the same range of $\arg Z(n\pi)$ as that for $\zeta_2>0$ with the difference $\arg Z(\kappa_1+n\pi)=(n-1)\pi+\pi/2$ and $\arg Z((n-1)\pi)=(n-1)\pi$, $n=-1, 0, 1, 2$. Thus we know from Eq. (6. 5) how θ_{21} depends on θ_1 and θ_2.

We are now in a position to know the sign of $b_{21}/(b_1b_2)$ in the (θ_1, θ_2)-plane, and thereby able to show by virtue of Eq. (3. 10) that the integral transforms under consideration give a representation of $SL_2(2, \boldsymbol{R})$. Let

(6. 6) $$\tan\chi_k=-\zeta_k e^{2\tau_k}, \quad k=1, 2, 21,$$

where $|\chi_k|<\pi/2$ for $k=1, 2$. For $k=21$, χ_k will be shortly determined uniquely. For $(\chi_k, \tau_k, \zeta_k)$, b_k vanish, $k=1, 2, 21$. Since one has $b_{21}=0$ when $b_1=b_2=0$, Eq. (6. 5) gives

(6. 7) $$\chi_{21}=\chi_2+\arg Z(\chi_1) \quad (\text{mod. } 4\pi),$$

by which χ_{21} is determined uniquely. To identify the sign of $b_{21}/(b_1b_2)$, we draw orthogonal lines $\theta_1=\chi_1+n\pi$ and $\theta_2=\chi_2+m\pi$ in the (θ_1, θ_2)-plane, n and m being integers. We next draw the curves defined by $\theta_{21}=\chi_{21}+k\pi$, k being integers, which pass the intersection points of the already drawn orthogonal lines, because at those points $b_1=b_2=b_{21}=0$. For example, the curve $\theta_{21}=\chi_{21}$ passes the points $(\chi_1+l\pi, \chi_2-l\pi)$, l being integers, and so on. Thus the (θ_1, θ_2)-plane is broken up into curved triangles, which are deformed from the triangles we considered in Sec. 4. The sign of $b_{21}/(b_1b_2)$ can now be definitely settled in each curved triangle.

We are ready to describe the composition of the integral transforms. Let U_k, $k=1, 2, 21$, be the integral transforms with certain values of β

which are determined by (6.4) with χ_k replaced for χ_0. The operators $\pm U_k$ are in two-to-one correspondence with M_k. Now we can prove, by the use of (3.10),

$$U_2U_1=U_{21}. \tag{6.8}$$

For example, in the region such that $\chi_1-\pi<\theta_1<\chi_1$, $\chi_2-\pi<\theta_2<\chi_2$, and $\chi_{21}-2\pi<\theta_{21}<\chi_{21}-\pi$, $b_{21}/(b_1b_2)$ is positive and moroovor, from (6.1) and (6.4) with χ_k replaced for χ_0, $\alpha_1=\alpha_2=ie^{-i\pi/4}$ and $\alpha_{21}=-e^{-i\pi/4}$. Then we have from (3.10)

$$U_1U_2=\alpha_1\alpha_2\alpha_{21}{}^{-1}e^{i\pi/4}U_{21}=U_{21}. \tag{6.9}$$

In other regions the same reasoning holds. Thus we have proved (6.8) for $b_1\neq 0$, $b_2\neq 0$.

We are in a final stage in studying canonical transforms. So far we have defined integral transforms for (θ, τ, ζ) such that $\theta\neq\chi_0+n\pi$, n being integers (see (6.3), (6.4)). The last task we are left with is then to define canonical transforms for $\theta=\chi_0+n\pi$. Definition is made by finding the limits of the integral kernels as θ tends to $\chi_0+n\pi$.

We take the integral kernel (3.5) again to consider its limit as $b\rightarrow 0$. Since $ad-bc=1$, we may understand that $a\neq 0$ when b is near to zero. Hence the integral kernel can be written, by replacing d with $(1+bc)/a$, in the form

$$K(x, y)=\alpha(2\pi|b|)^{-1/2}\exp\{(ic/2b)x^2\} \\ \exp\{(ia/2b)(y-x/a)^2\}. \tag{6.10}$$

It follows from (6.10) that, as $b\rightarrow 0$,

$$K(x, y)dy\rightarrow \begin{cases}\alpha_0(|a_0|)^{-1/2}e^{i\pi/4}\exp\left(\dfrac{ic_0}{2a_0}x^2\right) & \text{for } a/b>0,\\ \alpha_0(|a_0|)^{-1/2}e^{-i\pi/4}\exp\left(\dfrac{ic_0}{2a_0}x^2\right) & \text{for } a/b<0,\end{cases} \tag{6.11}$$

where the subscript 0 indicates the limit as $b\rightarrow 0$.

We are going to the limit of the kernel (6.10) as $\theta\rightarrow\chi_0$. The condition $|\chi_0|<\pi/2$ means that $a_0=e^{\tau}\cos\chi_0>0$, so that $a/b>0$ for $\theta\rightarrow\chi_0+0$, and $a/b<0$ for $\theta\rightarrow\chi_0-0$. Thus from (6.1) and (6.4), α tends to $\alpha_0=e^{-i\pi/4}$ or $\alpha_0=ie^{-i\pi/4}$ according as $\theta\rightarrow\chi_0+0$ or $\theta\rightarrow\chi_0-0$. Hence the expression

(6. 11) shows that, as $\theta\to\chi_0+0$

$$(6.12)\quad \int K(x,y)dy\to(|e^{\tau}\cos\chi_0|)^{-1/2}\exp\{(-i/2)x^2\tan\chi_0\},$$

so that we have from (6. 10) and (6. 12), as $\theta\to\chi_0$,

$$(6.13)\quad K(x,y)\to\exp(-i/2x^2\tan\chi_0)(|e^{\tau}\cos\chi_0|)^{-1/2}\delta((e^{\tau}\cos\chi_0)^{-1}x-y),$$

When $\tau=\zeta=0$, one has $\chi_0=0$, so that Expression (6. 13) reduces to $K(x,y)\to\delta(x-y)$, which we have already obtained in (4. 2).

In the same manner we obtain the limits of $K(x,y)$ as $\theta\to\chi_0+n\pi$, n being integers. We have indeed

$$(6.14)\quad K(x,y)\to i^k\exp[(-i/2)x^2\tan\chi_0](|e^{\tau}\cos\chi_0|)^{-1/2}\delta((-1)^k(e^{\tau}\cos\chi_0)^{-1}x-y)$$

as $\theta\to\chi_0-k\pi$, $k=-2,-1,0,1$. Expression (6. 14) reduces to (4. 9) when $\tau=\zeta=0$. When $\zeta=0$, one has $\chi_0=0$. Then expression (6. 14) then reads

$$(6.15)\quad K(x,y)\to i^k e^{-\tau/2}\delta((-1)^k e^{-\tau}x-y).$$

If $\theta\to 0$, Expression (6. 15) with $k=0$ reduces to (5. 6). However, if we let tend to 2π, (6. 15) with $k=-2$ becomes

$$(6.16)\quad K(x,y)\to -e^{-\tau/2}\delta(e^{-\tau}x-y).$$

This means that for the matrix $\exp(2\tau e_2)$, there are two canonical transforms $e^{-i\tau D}$ and $-e^{-i\tau D}$, which verifies that the canonical transforms give a realization of $SL_2(2,\boldsymbol{R})$.

Thus we have defined the unitary operators U for all parameters of $SL_2(2,\boldsymbol{R})$. The composition of operators $U_2U_1=U_{21}$ is true for all parameters of $SL_2(2,\boldsymbol{R})$ because of the continuity of U.

In conclusion, we point out that the canonical transformations we have constructed have two invariant closed subspaces, which are the space of even functions and of odd functions. To see this is an easy matter.

7. Canonical Transforms Representing SL(2, R)

In preceding sections we have proved that the quantization by the Schrodinger procedure give rise to a representation of $SL_2(2,\boldsymbol{R})$ in $L^2(\boldsymbol{R})$. In this section we wish to remark that the quantization in $L^2(\boldsymbol{R}^2)$

will provide us with a unitary representation of SL(2, $\boldsymbol{R}$) which is a direct sum of unitary irreducible representations in the space of functions of the radical variable. Though this fact is already claimed [19], we make a review of it from the viewpoint of reducing a quantum system. We will also answer a question as to why SL(2, $\boldsymbol{R}$) is represented, or no covering group of SL(2, $\boldsymbol{R}$) appear.

We begin with a classical system as in Sec. 2. Let $(\boldsymbol{R}^4, \omega)$ be a standard symplectic vector space endowed with the symplectic form

$$\omega = dp_1 \wedge dx_1 + dp_2 \wedge dx_2, \tag{7.1}$$

where (x_k, p_k), $k=1, 2$, are Cartesian coordinates. Consider the subgroup SO(2) $\times$ SL(2, $\boldsymbol{R}$) of SP(4, $\boldsymbol{R}$) given by

$$\begin{bmatrix} \exp(tN) & 0 \\ 0 & \exp(tN) \end{bmatrix} \begin{bmatrix} aI_2 & bI_2 \\ cI_2 & dI_2 \end{bmatrix}$$

with

$$N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad ad-bc=1. \tag{7.2}$$

Les one-parameter subgroups of SL(2, $\boldsymbol{R}$) act on $\boldsymbol{R}^4$. Then in the same manner as in Sec. 2, we have Hamiltonian function W_j, $j=1, 2, 3$, satisfying the sl(2, $\boldsymbol{R}$) commutation relations under the Poisson bracket,

$$\begin{aligned} W_1 &= 1/4(\langle p, p\rangle - \langle x, x\rangle), \\ W_2 &= 1/4(\langle x, p\rangle + \langle p, x\rangle), \\ W_3 &= 1/4(\langle p, p\rangle + \langle x, x\rangle). \end{aligned} \tag{7.3}$$

where $\langle\ ,\ \rangle$ stands for the standard inner product in $\boldsymbol{R}^2$, and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$. From the action of SO(2) we have

$$W_0 = 1/2\langle p, Nx\rangle, \tag{7.4}$$

the angular momentum.

We turn to a quantum system on $L^2(\boldsymbol{R}^2)$. By J_k, $k=1, 2, 3$, we mean the quantized operators associated with W_k, $k=1, 2, 3$, by the Schrodinger procedure $(p_j \to -i\partial/\partial x_j)$. Further, we introduce oscillator annihilation and creation operators by

$$a_k = -i/\sqrt{2}\,(x_k + \partial/\partial x_k), \quad a_k^+ = i/\sqrt{2}\,(x_k - \partial/\partial x_k), \quad k=1, 2. \tag{7.5}$$

Then J_k take the form

$$J=1/4\{(a_1^+)^2+(a_1)^2+(a_2^+)^2+(a_2)^2\},\quad J_2=1/(4i)\{(a_1^+)^2-(a_1)^2+(a_2^+)^2-(a_2)^2\},\quad J_3=1/2\{a_1^+a_1+a_2^+a_2+1\}. \tag{7.6}$$

We further mean by $J_0-1/2$ the quantized operator for W_0, half of the angular momentum, which has the form

$$J_0-1/2=1/(2i)(a_1^+a_2-a_2^+a_1). \tag{7.7}$$

The Casimir operator Q is defined and related to J_0 by

$$Q=(J_1)^2+(J_2)^2-(J_2)^2=J_0(1-J_0). \tag{7.8}$$

The operators J_k, $k=0, 1, 2, 3$, were employed in constructing oscillator realization of $\mathrm{SL}(2, \boldsymbol{R})$ in Ref. 19.

To consider the problem of covering, we deal with the operator J_3 related with the compact subgroup SO(2) of $\mathrm{SL}(2, \boldsymbol{R})$. Let $|m, n\rangle$ denote the normalized eigenfunctions of the harmonic oscillator, the double of J_3. Then we have in a familiar way

$$|m;n\rangle=(m!n!)^{-1/2}(a_1^+)^m(a_2^+)^n|0\rangle,\quad J_3|m;n\rangle=1/2(m+n+1)|m;n\rangle, \tag{7.9}$$

Where m and n are non-negative integers. In contrast with the one-dimensional case (2.14) the one-parameter group of unitary operators e^{-itJ_3} has the same period 4π as that of the corresponding classical one-parameter group;.

$$e^{-itJ_3}|m;n\rangle=e^{-1/2(m+n+1)t}|m;n\rangle,\quad \begin{pmatrix}x\\p\end{pmatrix}\to\begin{pmatrix}I_2\cos t/2 & I_2\sin t/2\\ -I_2\sin t/2 & I_2\cos t/2\end{pmatrix}\begin{pmatrix}x\\p\end{pmatrix}. \tag{7.10}$$

Thus we are not involved with the difficulty of covering group in the two dimensional case. In general, the covering group of $\mathrm{SL}(2, \boldsymbol{R})$ arises for odd-dimensional quantum systems, but does not for even-dimensional systems.

As is well known, oscillator realization (7.6) is reducible. Since the eigenvalues of the Casimir operator are prescribed by the eigenvalues j_0 of J_0 (see (7.8)), eigenspaces of J_0 are invariant closed subspaces, where

j_0's are half integers. If we employ the polar coordinates (r, θ), each of them proves to be the space of functions of thee form $e^{i(2j_0-1)}g(r)$ with j_0 fixed. We note further that such spaces labeled by j_0 are identified with $L^2(\boldsymbol{R}^+;rdr)$, $\boldsymbol{R}^+$ denoting the positive real numbers, by

$$(7.11)\qquad \int \overline{e^{i(2j_0-1)\theta}g_1(r)}e^{i(2j_0-1)\theta}g_2(r)rdrd \\ =2\pi\int_0^\infty g_1(r)g_2(r)rdr.$$

We may refer to this fact as the reduction of $L^2(\boldsymbol{R}^2)$ to $L^2(\boldsymbol{R}^+;rdr)$ by the S^1 action generated by the angular momentum $2J_0-1$.

In terms of representation, the realization of SL(2, $\boldsymbol{R}$) in $L^2(\boldsymbol{R}^2)$ by the Schrodinger quantization procedure is broken up into a series of unitary irreducible representations of positive type discrete classes $D_j{}^+$. $D_{1/2}{}^+$ appears once and $D_j{}^+$, $J_0=1, 3/2, \cdots$, twice each [19].

For the sake of consistency, we are to reproduce the oscillator realization of SL(2, $\boldsymbol{R}$) in $L^2(\boldsymbol{R}^+;rdr)$ in the form of integral transforms [2, 6, 19]. However, we would like to claim that we are strict in the phase factor of the integral kernel. Let SL(2, $\boldsymbol{R}$) be given in the form (7.2). A canonical transform U is then determined in the same manner as in preceding sections. The equation

$$(7.12)\qquad U\begin{pmatrix} x \\ p \end{pmatrix}U^{-1}=\begin{pmatrix} dI_2 & -bI_2 \\ -cI_2 & aI_2 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

is integrated to give, for $b\neq 0$,

$$Uf(x)=\int K(x, p)f(p)dp,$$

with

$$(7.13)\qquad K(x, p)=\alpha^2(2\pi|b|)^{-1}\exp[i/(2b)(d\langle x, x\rangle \\ -2\langle x, p\rangle+a\langle p, q\rangle)],$$

where $dy=dy_1dy_2$. The constant factor α^2 must be the square of the for the one-dimensional case, as easily seen from the matrix form of SL(2, $\boldsymbol{R}$) (see (7.12)). Thus we have $\alpha^2=e^{-i\pi/2}\beta^2$ form (6.1). Since $\beta^2=\text{sgn } b$ by (6.4), one has

$$(7.14)\qquad \alpha^2=-(\text{sgn } b),$$

which shows that no covering groups of $SL(2, \boldsymbol{R})$ appear.

If we adopt the polar coordinates by

$$(7.15) \qquad x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \\ y_1 = \rho\cos\phi, \quad y_2 = \rho\sin\phi,$$

the integral kernel (7.13) is expanded in a Fourier series,

$$(7.16) \quad K(x, y) = -i(\operatorname{sgn} b)(2\pi|b|)^{-1}\exp[i/(2b)(dr^2 + a\rho^2)] \\ X\sum_{m=-8}^{\infty}(-i)^m J_m(r\rho/b)e^{im\,\theta-\phi)}.$$

For functions $e^{i(2J_0-1)\theta}g(r)$, the integral transform (7.13) reduces, for each j_0, to the integral transform

$$(7.17) \qquad \int_0^\infty L_{j_0}(r, \rho)g(\rho)\rho d\rho,$$

where

$$L_{j_0}(r, \rho) = b_1 e^{i\pi j_0}\exp[i/(2b)(dr^2 + a\rho^2)]J_{2j_0-1}(r\rho/b).$$

Thus we have unitary irreducible representations D_j^+ of $SL(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R}^+; rdr)$.

The results (7.17) are essentially due to Mukunda and Radhakrishnan [19], and related to the results due to Moshinsky, Wolf, et al. [2, 4, 6] by the transformation $g(r) \to r^{1/2}g(r)$.

For one parameter subgroups of $SL(2, \boldsymbol{R})$, the infinitesimal transformations of (7.17) are obtained from J_k, $k=1, 2, 3$, by restricting them to the subspace of functions $e^{i(2j_0-1)\theta}g(r)$. One has in effect

$$(7.18) \quad J_1^r = 1/4\{-\partial^2/\partial r^2 - 1/r\partial/\partial r - r^2 + 1/r^2(2j_0-1)^2\}, \\ J_2^r = 1/(2i)(r\partial/\partial r + 1), \\ J_3^r = 1/4\{-\partial^2/\partial r^2 - 1/r\partial/\partial r + r^2 + 1/r^2(2j_0-1)^2\},$$

$SL(2, \boldsymbol{R})$ is also regarded as a dynamical group of a radical free particle with the Hamiltonian $J_1^r + J_3^r$.

Besides the series of unitary irreducible representations $D_{j_0}^+$, the quantization can give rise to a series of unitary irreducible representations of (nonexeptional) continuous class C_q^0 and $C_q^{1/2}$, if a subgroup $SO(1, 1) \times SL(2, \boldsymbol{R})$ of $Sp(4, \boldsymbol{R})$ is taken into consideration [3, 6, 19]. See also Ref. 6 in which, though within Lie algebra representations, oscillator realizations of $\mathrm{SL}(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R})$ are coupled to give representations of $\mathrm{SL}(2, \boldsymbol{R})$ in $L^2(\boldsymbol{R}^2)$ belonging to discrete series and to non-exceptional continuous series.

In conclusion we make a remark on the action in a classical system, which corresponds to the $\mathrm{SL}(2, \boldsymbol{R})$ action (7. 17) on $L^2(R^+; rdr)$. What we need to do is to reduce the symplectic vector space $(\boldsymbol{R}^4, \omega)$ by the S^1 action which are described by the first factor matrix in (7. 2) [20]. Consider the manifold M determined by $2W_0 = p_\theta = l$, l being a non-zero constant. By the action of SO(2), the first factor matrix in (7. 2), one reduces M to the orbit manifold $M_R = M/\mathrm{SO}(2)$, which can be realized in $\boldsymbol{R}^3$ as one of two-sheeted 2-hyperboloid by the equation

$$(7.19) \qquad -W_1^2 + W_2^2 + W_3^2 = W_0^2 = l^2/4.$$

The orbit manifold M_R is endowed with coordinates $r > 0$ and p_r to be identified tepologically with the product space $\boldsymbol{R}^+ \times \boldsymbol{R}$, and carries the symplectic form $dr \wedge dp_r$. We note here that since SO(2) and $\mathrm{SL}(2, \boldsymbol{R})$ commute, $\mathrm{SL}(2, \boldsymbol{R})$ acts on M_R. Further, the M_R realized in $\boldsymbol{R}^3$ is identified with an adjoint orbit of $\mathrm{SL}(2, \boldsymbol{R})$ in the dual space of the Lie algebra $\mathrm{sl}(2, \boldsymbol{R})$, so that action of $\mathrm{SL}(2, \boldsymbol{R})$ reduces to that of $\mathrm{SL}(2, \boldsymbol{R})/Z_2 = \mathrm{SO}(1, 2)$, the identity component of the Lorentz group $O(1, 2)$. In other words, the symplectic group $\mathrm{SO}_0(1, 2)$ acts on the phase space M_R endowed with $dr \wedge dp_r$. The group $\mathrm{SO}_0(1, 2)$ is thought of as a dynamical group of Hamiltonian systems on M_R reduced from those on $\boldsymbol{R}^4$.

We summarize by saying that with reduced classical system $(M_r, dr \wedge dp_r)$ admitting the symplectic action of $\mathrm{SO}_0(1, 2)$ is associated the reduced quantum system on $L^2(R^+; rdr)$ admitting the unitary action of $\mathrm{SL}(2, \boldsymbol{R})$, a double covering of $\mathrm{SO}_0(1, 2)$. The classical system $(M_R, dr \wedge dp_r)$ is parameterized by the angular meantime $p_\theta = l$, so is the quantum system $L^2(\boldsymbol{R}^+; rdr)$ by the angular momentum $2J_0 - i = 2j_0 - 1$.

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References

1. M. Moshinsky and C. Quesne, J. Math. Phys. **12**, 1772(1971); **12**, 780(1971).
2. M. Moshinsky, T.H. Seligman, and K.B. Wolf, J. Math. Phys. **3**, 901(1972).
3. K.B. Wolf, J. Math. Phys. **15**, 1295 (1974); **15**, 2102(1974).
4. C.P. Boyer and K.B. Wolf, J. Math. Phys. **16**, 1493(1975).
5. K.B. Wolf, J. Math, Phys. **17**, 601(1976);**21**, 680(1980).
6. D. Basu and K.B. Wolf, J. Math. Phys. **23**, 189(1982); **24**, 78(1983).
7. K.B. Wolf: Integral Transforms in Science and Engineering (Mathematical concepts and Methods in Science and Engineering, Vol. **11**, Plenum, New York, 1979).
8. V. Bargmann, Ann. Math. **59**, 1(1954).
9. D. Shale, Trans. Amer. Math. Soc. **103**, 149(1962).
10. N.R. Wallach, *Symplectic Geometry and Fourier Analysis* (*Lie Groups; History Frontiers and Applications*, Vol. 5, Math. Sci. Press, Brookline, Mass., 1977).
11. G. Lion and M. Vergne, *The Weil Representation, Maslov Index and Theta Series*, Progress in Mathematics 6, Birhauser, Boston, 1980.
12. V. Guillemin and S. Sternberg, *Symplectic Techniques in physics*, Cambridge University Press, Cambridge, 1984.
13. G. Burdet, M. Perrin, and M. Perroud, Commun. Math. Phys. **58**, 241(1978).
14. M. Perroud, Helv. Phys. Acta. **50**, 232(1977).
15. V. Bargmann, Ann. Math. **48**, 568(1647).
16. C. Ityzykson, Commun. Math. Phys. **4**, 92(1967).
17. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path integrals*, McGraw-Hill, New York, 1965.
18. J.M. Souriau, *Construction explicite de l'indice de Maslov*, Lecture Notes in Physics, **50**, 117, Springer-Verlag, New York, 1976.
19. N. Mukunda and B. Radhakrishnan, J. Math. Phys. **14**, 254(1973); **15**, 1320 (1974).
20. T. Iwai, J. Math. Phys. **26**, 885(1985).

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