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ON CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let S denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the disk $|z|<1$. A function f in S is said to be starlike of order $\alpha(0\leqq\alpha<1)$, denoted $f\in S^*(\alpha)$, if $\mathrm{Re}(zf'/f)>\alpha(|z|<1)$ and is said to be convex of order α, denoted $f\in K(\alpha)$, if $\mathrm{Re}(1+zf''/f')>\alpha$ $(|z|<1)$ [2].

Let T denote the subclass of S consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an analytic and univalent function f is in T if it can be expressed as

$$f(z)=z-\sum_{n=2}^{\infty} |a_n| z^n. \tag{1.1}$$

We also denote by $T^*(\alpha)$ and $C(\alpha)$ the subclasses of T that are, respectively, starlike of order α and convex of order α.

In this paper, we consider a subclass $A(\alpha,\lambda)$ $(0\leqq\alpha<1,\ 0\leqq\lambda\leqq1)$ of T. A function $f(z)$ of T is said to be in the class $A(\alpha,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} (n-\alpha)(n\lambda+1-\lambda)|a_n|\leqq 1-\alpha \tag{1.2}$$

for some α $(0\leqq\alpha<1)$ and for some λ $(0\leqq\lambda\leqq1)$.

The classes $A(\alpha,0)=T^*(\alpha)$ and $A(\alpha,1)=C(\alpha)$ was studied by Silverman [3].

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2. The classes $A(\alpha, \lambda)$

THEOREM 1. $A(\beta, \lambda) \subset A(\alpha, \lambda)$ *for* $0 \leqq \alpha \leqq \beta < 1$.

Proof. Let $\beta = \alpha + \delta$ for $\delta \geqq 0$. Then $f(z) \in A(\beta, \lambda)$ if and only if

$$(2.1) \qquad \sum_{n=2}^{\infty} \{n-(\alpha+\delta)\}(n\lambda+1-\lambda)|a_n| \leqq 1-(\alpha+\delta).$$

From (2.1), we have

$$(2.2) \qquad \sum_{n=2}^{\infty} (n\lambda+1-\lambda)|a_n| \leqq \frac{1-(\alpha+\delta)}{2-(\alpha+\delta)} < 1.$$

Therefore, if $f(z)$ is in $A(\beta, \lambda)$, then we have

$$(2.3) \qquad \begin{aligned} &\sum_{n=2}^{\infty} (n-\alpha)(n\lambda+1-\lambda)|a_n| \\ &\leqq 1-\alpha-\delta\Big(1-\sum_{n=2}^{\infty}(n\lambda+1-\lambda)|a_n|\Big) \\ &\leqq 1-\alpha, \end{aligned}$$

which implies $f(z) \in A(\alpha, \lambda)$.

THEOREM 2. $A(\alpha, \lambda_2) \subset A(\alpha, \lambda_1)$ *for* $0 \leqq \lambda_1 \leqq \lambda_2 \leqq 1$.

Proof. It is sufficient to show that

$$(2.4) \qquad n\lambda_1+1-\lambda_1 \leqq n\lambda_2+1-\lambda_2.$$

Since $\lambda_2 \geqq \lambda_1$, and $n \geqq 2$, we get

$$(2.5) \qquad (\lambda_2-\lambda_1)(n-1) \geqq 0.$$

If (2.5) is true, so is (2.4). The theorem is completed.

From Theorem 1 and Theorem 2, we have

THEOREM 3. $A(\alpha_2, \lambda_2) \subset A(\alpha_1, \lambda_1)$ *for* $0 \leqq \alpha_1 \leqq \alpha_2 < 1$ *and* $0 \leqq \lambda_1 \leqq \lambda_2 \leqq 1$.

3. Distortion and covering theorems

THEOREM 4. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $A(\alpha, \lambda)$, *then*

$$(3.1)\quad |z|-\frac{1-\alpha}{(1+\lambda)(2-\alpha)}|z|^2\leqq|f(z)|\leqq|z|+\frac{1-\alpha}{(1+\lambda)(2-\alpha)}|z|^2,$$

with the equality for the function

$$f(z)=z-\frac{1-\alpha}{(1+\lambda)(2-\alpha)}z^2.$$

Proof. Assume that $f(z)\in A(\alpha,\lambda)$. Then we have

$$(3.2)\quad (1+\lambda)(2-\alpha)\sum_{n=2}^{\infty}|a_n|\leqq\sum_{n=2}^{\infty}(n-\alpha)(n\lambda+1-\lambda)|a_n|\leqq 1-\alpha.$$

This gives

$$(3.3)\quad \sum_{n=2}^{\infty}|a_n|\leqq\frac{1-\alpha}{(1+\lambda)(2-\alpha)}.$$

Therefore

$$(3.4)\quad |f(z)|\leqq|z|+\sum_{n=2}^{\infty}|a_n||z|^n\leqq|z|+\frac{1-\alpha}{(1+\lambda)(2-\alpha)}|z|^2$$

and

$$|f(z)|\geqq|z|-\sum_{n=2}^{\infty}|a_n||z|^n\geqq|z|-\frac{1-\alpha}{(1+\lambda)(2-\alpha)}|z|^2.$$

Special cases of Theorem 5 has been proved by Silverman [3] $\lambda=0$ and $\lambda=1$ respectively.

From Theorem 4, we have

THEOREM 5. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $A(\alpha,\lambda)$, *then the disk* $|z|<1$ *mapped onto a domain that contains* $|w|<(\lambda(2-\alpha)+1)/((1+\lambda)(2-\alpha))$ *under* f. *This result is sharp with the extremal function*

$$f(z)=z-\frac{1-\alpha}{(1+\lambda)(2-\alpha)}z^2. \tag{3.5}$$

Special cases of Theorem 5 have been proved by Silverman [3], $\lambda=0$ and $\lambda=1$ respectively.

THEOREM 6. *If the function $f(z)$ defined by* (1.1) *is in the class $A(\alpha,\lambda)$, then*

$$1-\frac{2(1-\alpha)}{(1+\lambda)(2-\alpha)}|z|\leqq|f'(z)|\leqq 1+\frac{2(1-\alpha)}{(1+\lambda)(2-\alpha)}|z|. \tag{3.6}$$

The equality holds for the function

$$f(z)=z-\frac{1-\alpha}{(1+\lambda)(2-\alpha)}z^2.$$

Proof. We note that, for $f(z)\in A(\alpha,\lambda)$,

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|\leqq\frac{1-\alpha}{1+\lambda}. \tag{3.7}$$

It follows that from (3.3) and (3.7),

$$\begin{aligned}|f'(z)|&\leqq 1+\sum_{n=2}^{\infty}n|a_n||z|^{n-1}\\&\leqq 1+\sum_{n=2}^{\infty}n|a_n||z|\\&\leqq 1+\left(\frac{1-\alpha}{1+\lambda}+\alpha\sum_{n=2}^{\infty}|a_n|\right)|z|\\&\leqq 1+\frac{2(1-\alpha)}{(1+\lambda)(2-\alpha)}|z|.\end{aligned} \tag{3.8}$$

Using the same technique as above, we have

$$|f'(z)|\geqq 1-\frac{2(1-\alpha)}{(1+\lambda)(2-\alpha)}|z|. \tag{3.9}$$

Special cases of Theorem 6 has been proved by Silverman [3] $\lambda=0$ and $\lambda=1$ respectively.

4. Extreme points for $A(\alpha, \lambda)$

Brickman, MacGregor and Wilken [1] found the extreme points of the closed convex hull for convex, starlike, close to convex and typically real functions. Since then, the extreme points for many additional classes have been determined. We shall now determine the extreme points of $A(\alpha, \lambda)$.

THEOREM 7. *Let* $f_1(z)=z$ *and*

$$f_n(z)=z-\frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)}z^n \quad (n=2, 3, \cdots). \tag{4.1}$$

Then f is in the class $A(\alpha, \lambda)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z) \tag{4.2}$$

where $\lambda_n \geqq 0$ *and* $\sum_{n=1}^{\infty}\lambda_n=1$.

Proof. Suppose

$$\begin{aligned} f(z)&=\sum_{n=1}^{\infty}\lambda_n f_n(z) \\ &=z-\sum_{n=2}^{\infty}\lambda_n\frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)}z^n. \end{aligned} \tag{4.3}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty}\lambda_n\frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)}\cdot\frac{(n-\alpha)(n\lambda+1-\lambda)}{1-\alpha} \\ &=1-\lambda_1\leqq 1. \end{aligned} \tag{4.4}$$

It follows that $f\in A(\alpha, \lambda)$.

For the converse, suppose $f\in A(\alpha, \lambda)$. Since

$$|a_n|\leqq\frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)} \quad (n=2, 3, \cdots). \tag{4.5}$$

We may set

$$\lambda_n=\frac{(n-\alpha)(n\lambda+1-\lambda)}{1-\alpha}|a_n| \quad (n=2, 3, \cdots) \tag{4.6}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

This completes the proof.

COROLLARY 8. *The extreme points of* $A(\alpha, \lambda)$ *are given by* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)} z^n \quad (n=2, 3, \cdots) \tag{4.7}$$

Special cases of Corollary 8 has been proved by Silverman [3], $\lambda=0$ and $\lambda=1$ respectively.

5. Convexity theorems

THEOREM 9. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $A(\alpha, \lambda)$, *then* f *is convex in the disk*

$$|z| \leqq r(\alpha, \lambda) = \inf_{n \geqq 2} \left(\frac{(n-\alpha)(n\lambda+1-\lambda)}{n^2(1-\alpha)} \right)^{1/(n-1)} \tag{5.1}$$

The result is sharp, with the extremal function being of the form

$$f_n(z) = z - \frac{1-\alpha}{(n-\alpha)(n\lambda+1-\lambda)} z^n$$

for some n.

Proof. We note that

$$\left| z \frac{f''(z)}{f'(z)} \right| = \left| \frac{\sum_{n=2}^{\infty} n(n-1)|a_n| z^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n| z^{n-1}} \right| \tag{5.2}$$

$$\leqq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n|\,|z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1}}.$$

Thus

$$\left| z\frac{f''(z)}{f'(z)} \right| < 1 \text{ if}$$

(5.3) $$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} < 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1}$$

or

(5.4) $$\sum_{n=2}^{\infty} n^2|a_n||z|^{n-1} < 1.$$

Since $f \in A(\alpha, \lambda)$, (5.4) is true if

(5.5) $$n^2|z|^{n-1} < \frac{(n-\alpha)(n\lambda+1-\lambda)}{1-\alpha} \quad (n=2, 3, \cdots).$$

Solving (5.5) for z, we obtain

(5.6) $$|z| < \left(\frac{(n-\alpha)(n\lambda+1-\lambda)}{n^2(1-\alpha)}\right)^{1/(n-1)}.$$

Setting

$$r(\alpha, \lambda) = \inf_{n \geq 2} \left(\frac{(n-\alpha)(n\lambda+1-\lambda)}{n^2(1-\alpha)}\right)^{1/(n-1)},$$

the result follows.

Special cases of Theorem 9 have been proved by Silverman [3], $\lambda=0$.

References

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