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VALUE DISTRIBUTION OF AN ANALYTIC FUNCTION DOMINATED BY A SEQUENCE OF MONOMIALS

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1. Introduction

Let $f(z)$ be a meromorphic function defined on the complex plane $\boldsymbol{C}$. For any complex a, including ∞, we denote by

(1) $$n(r, a)=n(r, a, f)$$

the number of roots, of the equation $f(z)=a$ in $|z|\leq r$. We write

(2) $$N(r, a)=\int_0^r \frac{n(t, a)-n(0, a)}{t}dt+n(0, a)\log r.$$

If we assume that $f(0)\neq a$ then the formula (2) becomes siimply

(3) $$N(r, a)=\int_0^r \frac{n(t, a)}{t}dt.$$

Following Ahlfors [1] and Shimizu [3], we define

(4) $$A(r)=\frac{1}{\pi}\int_0^r t\,dt\int_0^{2\pi}\frac{|f'(te^{r\theta})|^2}{(1+|f|^2)^2}d\theta,$$

and define the Ahlfors and Shimizu characteristic by

(5) $$T(r)=\int_0^r \frac{A(t)}{t}dt.$$

We write

(6) $$M(r, a)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{[(1+|f(re^{i\theta})|^2)(1+|a|^2)]^{1/2}}{|f(re^{i\theta})-a|}d\theta.$$

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Under the assumption $f(z)\neq a$, one can write the first fundamental theorem in the form of Ahlfors and Shimizu;

$$T(r)=\int_0^r \frac{A(t)}{t}dt=N(r,a)+M(r,a)-M(0,a). \tag{7}$$

Let S be the Riemann sphere, a sphere of unit diameter lying on the complex plane and touching the plane at the origin. Consider the stereographic projection, if dp is an element of area in the plane near a point z and dA the corresponding element of area on the sphere, then $dA=dp/(1+|z|^2)$. Thus $\pi A(r)$ ((4) in the above) is the area (with counting multiplicity) of the image on the Riemann sphere of $|z|<r$ by $f(z)$.

By the formula (5), we have

$$r\frac{d}{dr}T(r)=A(r). \tag{8}$$

From (8), we know that if $f(z)$ is not constant in the plane then $T(r)$ is positive, convex, and strictly increasing function of log r. For a fixed a (7) shows that

$$N(r,a)<T(r)+0(1) \text{ as } r\to\infty. \tag{9}$$

From (9), one can deduce that

$$n(r,a)<A(r)+0(1) \text{ for a sequence of } r\to\infty. \tag{10}$$

If $f(z)$ is not constant, (10) shows that

$$\varliminf_{r\to\infty}\frac{n(r,a)}{A(r)}\leq 1.$$

Since

$$A(r)=\frac{\text{area of } f(\{z:|z|<r\})}{\text{area of Riemann sphere}}$$

we know that $A(r)$ is average number of times points on the Riemann sphere are covered by the image of $|z|<r$ by $f(z)$.

Let

$$n(r)=\sup_{a\in C\cup\infty} n(r,a).$$

We know that

$$1 \leq \frac{n(r)}{A(r)}.$$

Haymann and Stewart [3], proved that

$$1 \leq \varlimsup_{r\to\infty} \frac{n(r)}{A(r)} \leq e(=2.71\cdots).$$

In this paper we construct some functions, which give better estimation for the right hand side of the above inequality.

2. Main results

Let $f(z)$ be an analytic function on the complex plane. We adopt the following notations

(11) $$M(f, r) = \sup\{|f(z)| : z = r\}$$

and

(12) $$m(f, r) = \inf\{|f(z)| : z = r\}.$$

Through out this paper $\{r_k\}$ denote a positive strictly incresing sequence with $r_k \to \infty$ as $k \to \infty$.

Suppose that $f(z) = \sum_{n=0}^{\infty} f_n(z)$ where $f_n(z)$ are analytic and converges uniformly on compact subsets of the complex plane. Futher, for each r_k, we can choose an f_{n_k} such that

(13) $$\varlimsup_{k\to\infty} M(f(z) - f_{n_k}(z), r_k) / m(f_{n_k}(z), r_k) = 0.$$

Then we call $\{f_{n_k}, r_k\}$ dominates $f(z)$, and f_{n_k} dominates $f(z)$ at r_k. For the convenience of description, write

$$M_k = M(f_k, r_k), \qquad m_k = m(f_k, r_k)$$
$$\overline{M}_k = M(f - f_n, r_k), \qquad \overline{m}_k = m(f - f_k, r_k)$$

and

$$n\binom{R}{r} = \sup\{n(r, a) : |a| < R\}.$$

RROPOSITION 1. *Let $f(z)$ be analytic on the complex plane and it has a dominating sequence $\{f_k, r_k\}$ which satisfies*

$$m(f_{k+1}, r_{k+1}) \geq 2M(f_{k+1}, r_{k+1}).$$

Then

$$\varliminf_{r\to\infty} \frac{n\binom{R}{r}}{A(r)} \leq \varliminf_{k\to\infty} \frac{n\begin{Bmatrix}R\\ r_k\end{Bmatrix}}{A(r_k)} \leq \varlimsup_{k\to\infty} \frac{n(r_k, 0)}{A(r_k)}$$

for all R.

Proof. Let $f(0)=\beta$. Since, $m_k\to\infty$, one can choose a large k so that

$$(14) \qquad |\beta| < \frac{1}{3} m_k \ \text{ and } \ \overline{M}_k < \frac{1}{3} m_k.$$

By (14) and Rouche's theorem, we know that

$$f(z) - \beta = f(z) - f_k(z) + f_k(z) - \beta$$

and

$$f_k(z) - \beta$$

have the same number of zero's in $\{z : |z| < r\}$.
Hence, one can find η_1 and η_2 such that

$$f(\eta_1) = \beta, \quad |\eta_1| < r_k$$

and

$$f_k(\eta_2) = \beta, \quad |\eta_2| < r_k.$$

Set

$$\hat{M}_k = \sup_{|z|=r_k} \{|f(z)| : |z| = r_k\}$$

and

$$\hat{m}_k = \inf_{|z|=r_k} \{|f(z)| : |z| = r_k\}$$

If k is sufficiently large, (13) implies that

$$(15) \qquad m_{k-1} < \hat{m}_k < \hat{M}_k < m_{k+1}$$

Consider the following counting functions

$$(16) \qquad n(f, r_k, w) = \frac{1}{2\pi i}\int_{|z|=r_k} \frac{f'(z)}{f(z)-a}dz$$

and

$$(17) \qquad n(f_k, r_k, w) = \frac{1}{2\pi i}\int_{|z|=r_k} \frac{f_k'(z)}{f_k(z)-w}dz.$$

Since, $n(f, r_k, w)$ is integer valued and continuous on $\{w : |w| < \hat{m}_k\}$, it follows that

$$(18) \qquad n(f, r_k, w) = n(f, r_k, 0) = n(f, r_k, \beta)$$

for $|w| < \hat{m}_k$.

By the same argument, we have

$$(19) \qquad n(f_k, r_k, w) = n(f_k, r_k, 0) = n(f_k, r_k, \beta)$$

for $|w| < m_k$.

(18) and (19) show that

$$(20) \qquad n(f, r_k, w) = n(f_k, r_k, w) = n(f_k, r_k, 0) = n(f, r_k, 0)$$

for all $|w| < m_{k-1}$.

For each fixed $R (>0)$, we can choose an m_{k-1} so that

$$R < m_{k-1}.$$

Then, (10) shows that

$$\begin{aligned} n\cdot\begin{Bmatrix} R \\ r_k \end{Bmatrix} &= \sup\{n(r_k, w) : |w| < R\} \\ &= n(f, r_k, 0) \\ &= n(r_k, 0), \end{aligned}$$

and this proves the proposition.

Now we add another stric restriction on the dominating sequence of f, and prove the following.

PROPOSJTION 2. *Let f satitfy all of the conditions of the above proposition. Futher, assume that there is a finite constant M such that*

$$(21) \qquad n(f_{k+1}, r_{k+1}, \alpha) - n(f_k, r_k, \alpha) \leq M$$

for a fixed a and for all $k=1, 2, \cdots,$.
Then

$$\lim_{r\to\infty}\frac{n(r)}{A(r)}=1.$$

Proof. Let $|\lambda|<\hat{M}_n$. Then from (15), we have

$$|\lambda|<m_{k+1}<\hat{m}_{k+2}.$$

From (20), we know that

(22) $$n(f, r_k, \lambda)<n(f, r_{k+2}, \lambda)=n(f_{k+2}, r_{k+2}, \lambda)$$
$$=n(f, r_{k+2}, 0)=n(f_{k+2}, r_{k+2}, 0).$$

Choose k so large that $|\alpha|<\hat{m}_{k-1}$. Then from (20), (21) and (22), we have

$$n(f_k, r_k, \alpha)=n(f_k, r_k, 0)=n(f, r_k, 0)$$
$$n(f_{k+2}, r_{k+2}, \alpha)=n(f_{k+2}, r_{+2}, 0)=n(f, r_{k+2}, 0)$$

and

(23) $$n(f, r_k, \lambda)\leq n(f, r_k, 0)+2M$$

for all $|\lambda|\leq\hat{M}_k<m_{k+1}$.
Without loss of generality, we assume that $n(f, r)\to\infty$ (as $r\to\infty$). Hence, we have

(24) $$\lim_{k\to\infty} n(f, r_k, 0)=\infty.$$

Let $\varepsilon>0$, then by (23) and (24), we know that there is a k_0 such that

(25) $$(1+\varepsilon/2)n(f, r_k, 0)>n(f, r_k, \lambda)$$

for all $k\geq k_0$ and all $\lambda\in \boldsymbol{C}$.
It follows that

(26) $$(1+\varepsilon/2)n(f, r_k, 0)\geq n(f, r_k)=n(r_k).$$

Let $r\in[r_k, r_{k+1}]$, then

(27) $$n(r_k)\leq n(r)\leq n(r_{k+1})$$

and

$$(28) \qquad n(r_k, 0) \leq n(r, 0) \leq n(r_{k+1}, 0).$$

On the other hand (21) gives

$$n(r_{k+1}, 0) - n(r_k, 0) \leq M$$

for sufficiently large k.

Now choose $k_1 (\geq k_0)$ such that

$$(\varepsilon/2)\ n(r_k, 0) > 3M$$

for all $k \geq k_1$, then using (27) and (28), we conclude that

$$(29) \qquad (1+\varepsilon) n(r, 0) \geq n(r) \text{ for all } r \geq r_{k_1}.$$

Finally, we have

$$\begin{aligned} \varlimsup_{r\to\infty} \frac{n(r)}{A(r)} &\leq \varlimsup_{r\to\infty} \frac{(1+\varepsilon) n(r, 0)}{A(r)} \\ &\leq (1+\varepsilon) \varlimsup_{r\to\infty} \frac{n(r, 0)}{A(r)} \\ &\leq (1+\varepsilon). \end{aligned}$$

Since ($\varepsilon > 0$) is arbitrally constant, it gives

$$\lim_{r\to\infty} \frac{n(r)}{A(r)} = 1.$$

Now we want to construct an analytic function with dominating sequences. In [2], [Arias] constructed an f which has exceptional sets of the second fundamental theorem of Nevalina theory. The function admits a dominating sequences.

Following his method, we construct an example.

EXAMPLE. Let $\{\lambda_n\}$ be a positive integer valued sequence satisfying $\lambda(n) > 2\lambda(n-1)$ and $\{T_n\}$ be a positive sequence with $T_{n+1} \leq (T_n/2)^2$. Let A_n be given by

$$A_n = \begin{cases} (T_n)^n : \lambda(2k) \leq n \leq \lambda(2k+1) \\ 0 \quad\ : \lambda(2k-1) < n < \lambda(2k) \end{cases}$$

Then the analytic function

$$f(X)=\sum_{n=0}^{\infty}A_nX^n$$

admits a dominating sequence.

Proof. Set

$$\sum_1^k(X)=\sum_{n=0}^{\lambda(2k-1)}A_nX^n,$$

$$\sum_k^2(X)=A\lambda_{(2k)}X^{\lambda(2k)}=f_k(X),$$

and

$$\sum_2^k(X)=\sum_{n=\lambda(2k)+1}^{\infty}A_nX^n.$$

We shall show that (f_k, r_k) dominates f, where r_k is defined by

$$r_k=2/(T_{\lambda(2k)})^2.$$

Now we give an estimation of $\sum_i$;

$$\begin{aligned}\sum_2^k&=\{|f_k(X)| : |X|=r_k\}\\&=(T_{\lambda(2k)}\ r_k)^{\lambda(2k)}\\&=[T_{\lambda(2k)}\cdot 2(T_{\lambda(2k)})^{-2}]^{\lambda(2k)}\\&=2^{\lambda(2k)}/(T_{\lambda(2k)})^{\lambda(2k)},\end{aligned}$$

$$\begin{aligned}\sum_1^k&=\{|\sum_1^k(X)| : |X|=r_k\}\\&=\sum_{n=0}^{\lambda(2k-1)}A_n(2/(T_{\lambda(2k)})^2)^n\\&\leq\lambda(2k-1)(2(T_{\lambda(2k)})^2)^{\lambda(2k-1)}\\&=\lambda(2k-1)2^{\lambda(2k-1)}/(T_{\lambda(2k)})^{\lambda(2k-1)}.\end{aligned}$$

By a simple calculation, we have

$$\sum_1^k/\sum_2^k\leq\lambda(2k-1)/2^{\lambda(2k-1)}$$

and

$$\lim_{k\to\infty}\sum_1^k/\sum_2^k\le\lim_{l\to\infty}\frac{l}{2^l}=0.$$

For the estimation of $\sum_3^k$, consider

$$\begin{aligned}&A_{\lambda(2k)+l}X^{\lambda(2k)+l}|_{X=r_k}\\&=(T_{\lambda(2k)+l})^{\lambda(2k)+l}\left(\frac{2}{T_{\lambda(2k)+l}}\right)^{\lambda(2k)+l}\\&=(T_{\lambda(2k)}2^l)^{\lambda(2k)+l}(2^{-2^l})^{\lambda(2k)+l}[2^{\lambda(2k)+l}/(T_{\lambda(2k)})^{\lambda(2k)+l}]\\&\cdot 2^{-2l(\lambda(2k)+l)}\\&\le 2^{-\lambda(2k)+l}.\end{aligned}$$

Hence, we have

$$\sum_3^k=\{|\sum_3^k(X)|:X=r_k\}$$

$$\le\sum_{n=0}^{\infty}2^{-n}=2.$$

From the above calculation, we have

$$\lim_{k\to\infty}\overline{M}_k/m_k=0$$

and

$$m_{k+1}=M_{k+1}>2M_k.$$

Hence we know that $\{f_k, r_k\}$ dominates f. But it does not satisfy (21).

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