Comm. Korean Math. Soc. 2 (1987), No. 2, pp. 151~160

SYMBOLIC CALCULUS OF HERMITE OPERATORS

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1. Introduction

In this paper, we are concerned with the study of symbolic calculus of so-called Hermite operators which first defined by Grushin[7] and later used (in a refined form) quite successfully by Boutet de Monvel and Trèves in [2, 3]. Hermite operators are going to be continuous linear mappings from $C_0^\infty(R^n)$ to $C^\infty(R^{n+1})$ (cf. Definition 2.3). However, they are neither pseudo-differential operators nor Fourier integral operators.

If P(or Q) is a pseudo-differential operator on R^{n+1} (or on R^n) and H is a Hermiter operator on R^n, then we can compose them as $P\circ H$(or $H\circ Q$) which yields another Hermite operator on R^n. We can also take a conjugation of P by a Hermite operator like H^*PH, where H^* denotes the adjoint of H in L^2-sense, which possibly reduces the study of pseudo-differential operator P on R^{n+1} to that of another pseudo-differential operator H^*PH on R^n. In doing all these, we need to develop certain symbolic calculus connecting them, which is, in fact, our main concern in this paper. We will define Hermite operators by a different way from that in [2] and show some basic continuity properties of Hermite operators on function or distribution spaces. And we will show that our definition is essentially equivallent to that of Boutet de Monvel and Trèves[2] and formulate some symbolic calculus between Hermite operators and pseudo-differential operators. Finally we will consider an application of Hermite operators.

2. Definition and Continuity of Hermite Operators

In this and the next section, let us set $\Omega'=R^n$ and $\Omega=R^{n+1}$. The variable in Ω will be denoted by (x, t), $x=(x_1, \cdots, x_n)$; the dual variable

Received July 30, 1987.
Supported, in part, by Korea Science and Engineering Foundation.

will be denoted by (ξ, τ), $\xi=(\xi_1, \cdots, \xi_n)$. Like a pseudo-differential operator, a Hermite operator will be defined as an oscillating integral with amplitude $a(x, t, y, \xi) \in C^\infty(\Omega\times\Omega'\times R^n)$ satisfying a certain growth condition. More precisely, we have the following which is a modification of what has been done in [2].

DEFINITION 2.1. Let m be any real number. We shall denote by h^m the linear space of C^∞-functions $a(x, t, y, \xi)$ in $\Omega\times\Omega'\times R^n$, which have the following property: Given any pair of integers $j, k\geq 0$, any triplet of n-tuples α, β, γ, and any compact subset K of $\Omega\times\Omega'$, there is a constant $C>0$ such that

$$|t^j\partial_t^k\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma a(x, t, y, \xi)|\leq C(1+|\xi|)^{m-|\alpha|-(j-k)/2}$$

for all $(x, t, y)\in K$, $\xi\in R_n$.

EXAMPLE. ([2]) Let $\mathscr{H}_j(u)=(2^j j!)^{1/2}\left(\frac{\partial}{\partial u}-u\right)^j e^{-u^2/2}$ is the j-th Hermite function. Then, $a_j(x, t, \xi)=\left(\frac{|\xi|}{\pi}\right)^{1/4}\mathscr{H}_j(t|\xi|^{1/2})$ is in $h^{1/4}$ for each $j=0, 1, 2, \cdots$.

h^m is a Fréchet space when it is equipped with the topology defined by semi-norms:

$$P_{m,i,l}(a)=\sup(1+|\xi|)^{-m+|\alpha|+(j-k)/2}|t^j\partial_t^k\partial_\xi^\alpha\partial_x^\beta\partial_y^\gamma a(x, t, y, \xi)|,$$

where the supremum is taken over all the ranges such that $|\alpha|+|\beta|+|\gamma|+j+k\leq l$, $(x, t)\in K_i$ and K_i is an exhausting sequence of compact sets of Ω. It is clear that the inclusion map $h^m\to h^{m'}$ is continuous if $m\leq m'$.

We will call elements of h^m as Hermite-amplitudes of degree m. What we shall later introduce as Hermite-symbols will be Hermite-amplitudes that are independent of y. From now on we will omit the adverb "Hermite" since it will be clear from the context. For the (Hermite-) amplitudes, we also have the notion of asymptotic series expansion and the corresponding theorem and its proof is very similar to that of pseudo-differential operators ([1, 5]).

THEOREM 2.2. *Let (a_k) be a sequence of amplitudes with degree $m_k\downarrow -\infty$. Then there exists an amplitude a of degree m_0 such that $a\sim\Sigma a_k$*

We are ready to define Hermite operators.

DEFINITION 2.3. A Hermite operator H of degree m on Ω' is a linear operator $H : C_0^\infty(\Omega') \to C^\infty(\Omega)$ defined by an oscillating integral

$$Hf(x, t) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x, t, y, \xi) f(y) dy d\xi,$$

where $a \in h^m$. We will denote by OPH^m the space of Hermite operators of degree m. And let us set $OPH^{-\infty} = \bigcap_m OPH^m$

Although it is easy to see that Hermite operators map $C_0^\infty(\Omega')$ into $C^\infty(\Omega)$ continuously and can be extended as continuous linear mappings of $\mathcal{E}'(\Omega')$ into $D'(\Omega)$, we can get these as direct consequences of the continuity of Hermite operators on Sobolev spaces.

THEOREM 2.4. *Let $A \in OPH^m$. Given any real number s, $u \mapsto Au$ can be extended as a continuous linear mapping of $H_c^s(\Omega')$ into $H_{loc}^{s-m}(\Omega)$.*

Proof. It sufficies to show that for every function $g \in C_0^\infty(\Omega)$ and for every compact subset K of Ω', there is a constant $C > 0$ such that

$$\|gAu\|_{s-m} \leq C\|u\|_s, \quad u \in C_0^\infty(K),$$

where $\|u\|_s^2 = \int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$.

It is clear that nothing is changed if we replace gA by gAh with $h \in C_0^\infty(\Omega')$ equal to one in some neighborhood of K. Observe that an amplitude for gAh is given by $g(x, t) a(x, t, y, \xi) h(y)$, where $a(x, t, y, \xi)$ is an amplitude of A. Thus, from the beginning, we may assume that the (x, t, y)-projection of support of $a(x, t, y, \xi)$ is contained in a compact subset of $\Omega \times \Omega'$.

Let us denote by $\tilde{a}(\xi', \tau, \eta', \xi)$ the Fourier transform of $a(x, t, y, \xi)$ with respect to (x, t, y). We derive at once that to every triplet of integers $i, j, k \geq 0$, there is a constant $C > 0$ such that

$$(2.1) \quad |\tilde{a}(\xi', \tau, \eta, \xi)| \leq C(1+|\xi|)^m (1+|\xi'|)^{-i} (1+|r|)^{-j} (1+|\eta'|)^{-k}$$

for every $\xi', \xi, \eta' \in R_n$, $\tau \in R_1$. By virtrue of the Fourier inversion formula we have

$$(2.2) \quad \widehat{Au}(\xi', \tau) = (2\pi)^{-2n} \int \tilde{a}(\xi' - \xi, \tau, \xi - \eta', \xi) \hat{u}(\eta') d\xi d\eta'.$$

Multiplying both sides of (2.2) by $(1+|(\xi', \tau)|^2)^{(s-m)/2}$ and using (2.1), we get

$$(1+|(\xi', \tau)|^2)^{(s-m)/2} |\widehat{Au}(\xi', \tau)|$$

$$=(1+|\xi'|^2+|\tau|^2)^{(s-m)/2}|\widehat{Au}(\xi',\tau)|\leq C(1+|\tau|)^{-1}$$
$$\int(1+|\xi-\xi'|)^{-n-1}(1+|\xi-\eta'|)^{-n-1}(1+|\eta'|^2)^{s/2}|\hat{u}(\eta')|^2d\xi d\eta'.$$

Since $\|f*g*h\|_{L^2}\leq\|f\|_{L^1}\cdot\|g\|_{L^1}\cdot\|h\|_{L^2}$ we have

$$\|Au\|_{s-m}=\int(1+|(\xi',\tau)|^2)^{(s-m)}|\widehat{Au}(\xi',\tau)|^2d\xi'd\tau\leq C\|u\|_s.$$

The following corollaries are immediate consequences of Theorem 2.4.

COROLLARY 2.5. *Every Hermite operator A defines a continuous linear mapping of $C_0^\infty(\Omega')$ into $C^\infty(\Omega)$ which extends to a continuous linear mapping of $\mathcal{E}'(\Omega')$ into $D'(\Omega)$.*

COROLLARY 2.6. *Every Hermite operator A of degree $-\infty$ is regularizing i.e. it maps $\mathcal{E}'(\Omega')$ into $C^\infty(\Omega)$.*

As in the case of pseudo-differential operators, we also have the following theorem which will be used later([9]).

THEOREM 2.7. *If $A\in OPH^m$, then there exists another Hermite operator $A^\#$ of degree m such that $A-A^\#$ is regularizing and the (x,y)-projection of the support of the amplitude $A^\#$ lies in the neighborhood of the diagonal Δ in $\Omega'\times\Omega'$.*

3. Symbolic Calculus of Hermite Operators

Let m be any real number. We shall denote by S^m the subspace of h^m consisting of the amplitudes $a(x,t,\xi)$ which are independent of y. And let us set $S^{-\infty}=\bigcap_m S^m$, $m\in R^1$. Now the elements of the space S^m will be called the symbols of degree $\leq m$ in Ω'. We have defined Hermite operators using amplitudes in the previous section 3. But, in [2], they are defined by symbols. We shall prove that the two definitions are equivalent modulo regularizing operators.

THEOREM 3.1. *Let $A\in OPH^m$ with amplitude $a(x,t,y,\xi)$. Then there exists a symbol $\tilde{a}(x,t,\xi)\in S^m$ such that A is equivalent (modulo regularizing operators) to the (Hermite) operator $\tilde{A}$ defined by*

$$\tilde{A}f(x,t)=(2\pi)^{-n}\int e^{ix\cdot\xi}\tilde{a}(x,t,\xi)\hat{f}(\xi)d\xi,\quad f\in C_0^\infty(\Omega').$$

Moreover

$$\tilde{a}(x, t, \xi) \sim \Sigma \frac{i^{-|\alpha|}}{\alpha!} \partial_y{}^\alpha \partial_\xi{}^\alpha a(x, t, y, \xi)\,|_{y=x}.$$

Proof. By Theorem 2.7, we may assume that the (x, y)-projection of the support of $a(x, t, y, \xi)$, an amplitude of A, lies in the neighborhood of the diagonal Δ in $\Omega' \times \Omega'$ This enables us to write the finite Taylor expansion of $a(x, t, y, \xi)$ with respect to y in the following manner:

$$(3.1)\quad a(x, t, y, \xi) = \sum_{|\alpha| \le N} \frac{1}{\alpha!} (y-x)^\alpha \partial_y{}^\alpha a(x, t, x, \xi) + r_N(x, t, y, \xi),$$

where

$$\begin{aligned} r_N(x, t, y, \xi) &= \frac{1}{N!} \int_0^1 (1-s)^N \partial_s{}^{N+1} a(x, t, x+s(y-x), \xi)\, ds \\ &= (N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} (y-x)^\alpha \\ &\quad \int_0^1 (1-s)^N (\partial_y{}^\alpha a)(x, t, x+s(y-x), \xi)\, ds. \end{aligned}$$

We shall now consider an amplitude of the kind

$$(3.2)\quad (y-x)^\alpha b(x, t, y, \xi),$$

where $b(x, t, y, \xi) \in h^m$. Let B_α be a Hermite operator with amplitude $(y-x)^\alpha b(x, t, y, \xi)$. Then we have

$$\begin{aligned} B_\alpha u(x, t) &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} (y-x)^\alpha b(x, t, y, \xi) u(y)\, dy d\xi \\ &= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} D_\xi{}^\alpha b(x, t, y, \xi) u(y)\, dy d\xi. \end{aligned}$$

We reach the conclusion that another amplitude of the operator B_α is $D_\xi{}^\alpha b(x, t, y, \xi)$, which implies that $B_\alpha \in OPH^{m-|\alpha|}$. We apply these considerations to the terms in the finite Taylor expansion (3.1) and in particular to the remainder r_N. We reach the following conclusion: If $a(x, t, y, \xi) \in h^m$ is an amplitude of A, another amplitude of A is given by

$$\sum_{|\alpha| \le N} \frac{1}{\alpha!} D_\xi{}^\alpha D_y{}^\alpha a(x, t, x, \xi) + r_N{}^\#(x, t, y, \xi),$$

where

$$\begin{aligned} & r_N{}^\#(x, t, y, \xi) \\ &= (N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \int_0^1 (1-s)^N D_\xi{}^\alpha \partial_y{}^\alpha a(x, t, x+s(y-x), \xi)\, ds. \end{aligned}$$

Let us denote by $A_\alpha^\#$ (resp $R_N^\#$) a Hermite operator with amplitude $D_\xi^\alpha\partial_y^\alpha a(x,t,x,\xi)$ (resp. $r_N^\#(x,t,y,\xi)$). We have just shown that

$$(3.3) \qquad A=\sum_{|\alpha|\le N}\frac{1}{\alpha!}A_\alpha^\# + R_N^\# \quad mod\ OPH^{-\infty}.$$

It immediately follows that

$$(3.4) \qquad D_\xi^\alpha\partial_y^\alpha a(x,t,x,\xi)\in S^{m-|\alpha|}$$

and

$$(3.5) \qquad r_N^\#(x,t,y,\xi)\in h^{m-(N+1)}.$$

Consequently we have $A_\alpha^\#\in OPH^{m-|\alpha|}$ and $R_N^\#\in OPH^{m-(N+1)}$. Another consequence of (3.4) and (3.5) is that the formal series

$$(3.6) \qquad \sum\frac{1}{\alpha!}D_\xi^\alpha\partial_y^\alpha a(x,t,x,\xi)$$

defines a symbol $\tilde{a}(x,t,\xi)\in S^m$ of a Hermite operator. Then formula (3.3) proves what we want.

Now we wish to mention the composition laws (modulo regularizing operators) for Hermite operators and pseudo-differential operators. Their proofs are similar to those for pseudo-differential operators ([1, 2, 9]). In the sequel, $\underline{\sigma}(A)$ denotes the total symbol of an operator A.

THEOREM 3.2. *If K is a Hermite operator of degree m with symbol $k(x,t,\xi)$ and if B is a pseudo-differential operator of degree m' with symbol $b(x,\xi)$, then $K\circ B$ is a Hermite operator of degree $m+m'$ with symbol defined by*

$$\sum_\alpha\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha k(x,t,\xi)\partial_x^\alpha b(x,\xi).$$

Proof. We may assume that $\phi(x,y)b(x,\xi)$ is an amplitude of B, where ϕ is properly supported and equal to one in a neighborhood of the diagonal Δ in $\Omega'\times\Omega'$ Then we can compose $K\circ B$ and obtain that

$$K\circ Bu(x,t)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}h(x,t,y,\xi)u(y)dyd\xi.$$

where

$$h(x,t,y,\xi)=k(x,t,\xi)(2\pi)^{-n}\int e^{i(y-z)\cdot(\xi-\eta)}b(z,\eta)\phi(z,y)dzd\eta.$$

By taking the Taylor expansion of $b(z,\eta)\phi(z,\eta)$ with respect to (z,η) about (y,ξ), we obtain that

$$h(x,t,y,\xi)\sim k(x,t,\xi)\sum\frac{i^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\partial_z^\alpha b(z,\xi)\phi(z,y)\,|_{z=y}$$
$$=k(x,t,\xi)\sum\frac{(-1)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\partial_y^\alpha b(y,\xi).$$

Thus we have

$$\underline{\sigma}(KB)=\sum_\beta\frac{i^{-|\beta|}}{\beta!}\{k(x,t,\xi)\sum\frac{i^{|\alpha|}}{\alpha!}\partial_\xi^\alpha\partial_y^\alpha b(y,\xi)\}\,|_{y=x}$$
$$=\sum_\gamma\frac{i^{-|\gamma|}}{\gamma!}\partial_x^\gamma b(x,\xi)\partial_\xi^\gamma k(x,t,\xi).$$

THEOREM 3.3. *If A is a pseudo-differential operator of degree m in R^{n+1} with symbol $a(x,t,y,\xi)$ and if K is a Hermite operator of degree m' with symbol $k(x,t,\xi)$, then $A\circ K$ is a Hermite operator with symbol defined by*

$$\sum_{\alpha,p,q}\frac{i^{-|\alpha|-q}(-1)^{|q|}}{\alpha!q!}\{\partial_\xi^\alpha a_{p,q}(x,\xi)\}\{t^p\partial_t^q\partial_x^\alpha k(x,t,\xi)\}.$$

Proof. Let $\phi(x,t,y,s)$ and $\psi(x,y)$ be properly supported and equal to one in a neighborhood of the diagonal in $\Omega\times\Omega$ and $\Omega'\times\Omega'$, respectively. Then we can compose $A\circ K$ modulo regularizing operators and obtain that

$$A\circ Ku(x,t)=(2\pi)^{-n}\int e^{i(x-y)\cdot\xi}h(x,t,y,\xi)u(y)dyd\xi,$$

where

$$h(x,t,y,\xi)=(2\pi)^{-n}\int e^{(y-z)\cdot(\xi-\eta)+i(t-s)\tau}$$
$$a(x,t,\xi,\tau)\phi(x,t,z,s)k(z,\eta)\psi(x,y)dzd\eta dsd\tau.$$

Similarly, by taking Taylor expansion, we can obtain that

$$h(x,t,y,\xi)\sim\sum_{\alpha,q}\frac{i^{|\alpha|+q}}{\alpha!q!}\{\partial_\tau^q a(x,t,\xi,\tau)\,|_{\tau=0}\}$$
$$\partial_s^q\partial_y^\alpha\{\phi(x,t,y,s)\partial_\xi^\alpha k(y,s,\xi)\}\,|_{s=t}$$

Therefore we have

$$\underline{\sigma}(AK) \sim \sum_{\beta} \frac{(-i)^{|\beta|}}{\beta!} \partial_y^{\beta} \partial_\xi^{\beta} h(x, t, y, \xi)\,|_{y=x}$$
$$= \sum_{\gamma, p, q} \frac{i^{-|\gamma|-q}(-1)^q}{\gamma! q!} \{\partial_\xi^{\gamma} a_{p,q}(x, \xi)\}\{t^p \partial_t^q \partial_x^{\gamma} k(x, t, \xi)\},$$

where $\sum a_{p,q}(x, \xi) t^p \tau^q$ is the (formal) Taylor expansion of $a(x, t, \xi, \tau)$ about $t=\tau=0$.

THEOREM 3.4. *If K and K' are Hermite operators with symbol $k(x, t, \xi)$ and $k'(x, t, \xi)$, respectively, which vanish for (x, t) outside some compact set, then $K^* \circ K'$ is a pseudo-differential operator with symbol defined by*

$$\sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_x^{\alpha} \int_{-\infty}^{\infty} \{\partial_\xi^{\alpha} \overline{k(x, t, \xi)}\} k'(x, t, \xi) dt.$$

Proof. We may assume that $\phi(x, y) k(x, t, \xi)$ and $\psi(x, y) k'(x, t, \xi)$ are amplitudes for K and K', respectively, where ϕ and ψ are properly supported and equal to one in a neighborhood of the diagonal in $R^n \times R^n$. Then we can compose $K^* \circ K'$ and obtain that

$$K^* \circ K' u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} h(x, y, \xi) u(y) dy d\xi,$$

where

$$h(y, y, \xi)$$
$$= (2\pi)^{-n} \int e^{i(y-z)\cdot(\xi-\eta)} \overline{k(z, t, \xi)} \phi(z, x) k'(z, t, \eta) \psi(z, y) dz d\eta dt.$$

By taking Taylor expansion of $\overline{k(z, t, \xi)} \phi(z, x) k'(z, t, \eta) \psi(z, y)$ with respect to (z, η) about (y, ξ), we obtain that

$$h(x, y, \xi) = \int \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \partial_\eta^{\alpha} \partial_z^{\alpha} \{\overline{k(z, t, \xi)} \phi(z, x) k'(z, t, \eta) \psi(z, y)\}\,|_{\substack{z=y \\ \eta=\xi}}\, dt$$
$$= \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \int \partial_y^{\alpha} \{\overline{k(y, t, \xi)} \phi(y, x) \partial_\xi^{\alpha} k'(y, t, \xi)\} dt.$$

Therefore we have

$$\underline{\sigma}(K^* K') \sim \sum_{\beta} \frac{i^{-|\beta|}}{\beta!} \partial_y^{\beta} \partial_\xi^{\beta} h(x, y, \xi)\,|_{y=x}$$

$$=\sum_{\gamma}\frac{i^{-|\gamma|}}{\gamma!}\partial_x^{\gamma}\int k'(x,t,\xi)\partial_\xi^{\gamma}\overline{k(x,t,\xi)}dt.$$

4. An Application

Operators $L^{\pm}=D_t\mp it|D_x|$ are the prototype of operators with simple characteristics, where $|D_x|$ is the pseudo-differential operator with symbol $|\xi|$. As an application of Hermite operators, we will study the operators $L^{\pm}$. Note that $(L^+)^*=L^-$. Let H be a Hermite operator with symbol $e^{-t^2|\xi|/2}$, that is,

$$Hf(x,t)=(2\pi)^{-n}\int e^{ix\cdot\xi}e^{-t^2|\xi|/2}\hat{f}(\xi),d\xi,\quad f\in C_0^\infty(R^n).$$

By performing Fourier-transform with respect to x, we have $\widehat{L^+Hf}(\xi,t)=0$. Thus $L^+H=0$. Moreover $H^*H=(\pi/|D_x|)^{1/2}$ since the symbol of H^*H is

$$\int_{-\infty}^{\infty}e^{-t^2|\xi|/2}e^{-t^2|\xi|/2}dt=\sqrt{\pi/|\xi|}.$$

If we put $K=H(|D_x|/\pi)^{1/2}H^*$, then we have $KH=H, K^*=K, K^2=K$ and $L^+K=0$. Let us set

$$Ef(x,t)=(2\pi)^{-n}\iint_0^t ie^{ix\cdot\xi}e^{(-t^2+t'^2)|\xi|/2}\hat{f}(\xi,t')dt'd\xi.$$

We can check that the operator E is a right inverse of L^+(i.e. $L^+E=I$) and that E maps $S(R^{n+1})$ into itself and also $S'(R^{n+1})$ into itself. Since $L^+E=I$ we can say that L^+ maps $S(R^{n+1})$ onto itself and maps $S'(R^{n+1})$ onto itself.

Now let us set $F=(I-K)E$. Then F is also a right inverse of L^+ since $L^+F=L^+(I-K)E=L^+E-L^+KE=L^+E=I$. By taking adjoints, we have $F^*L^-=I$. Thus F^* is a left inverse of L^- and L^- is one to one from $S(R^{n+1})$ to itself and from $S'(R^{n+1})$ to itself.

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