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MULTIPLIERS IN SPACES OF HOLOMORPHIC FUNCTIONS

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Introduction

Let f be an analytic function on the unit disk. We shall use the standard notation,

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p},\ 0<p<\infty,$$
$$M_\infty(r,f)=\max_{0\leq\theta\leq 2\pi}|f(re^{i\theta})|.$$

The classes G^p, B^p and $D^{p,q}$ are defined as follows: $f\in G^p$ $(p>0)$ if and only if $\int_0^1 M_1(r,f')^p dr<\infty$, $f\in B^p$ $(0<p<1)$ if and only if $\int_0^1(1-r)^{(1/p)-2}M_1(r, f(dr<\infty$, $f\in D^{p,q}(0<p<\infty, 1\leq q<\infty)$ if and only if

$$\int_0^1(1-r)^{-(p/q)+p}M_q(r,f')^p dr<\infty.$$

P. Ahern and M. Jevtic ([1]) showed that for $0<p<q$,

$$D^{p,q}=\{f:\int_0^1(1-r)^{-p/q}M_q(r,f)^p dr<\infty\}.$$

They also showed that $D^{p,2}$ are exactly the spaces D^p introduced by F. Holland and B. Twomey ([7]). In [7], they showed that $B^p-D^p\neq\phi$ and $D^p-B^p\neq\phi$.

In the first section we show that $B^p\subset D^{s,q}$ for $0<s<p<1<q<\infty$. Therefore $B^p\subset D^s$ for all $0<s<p<1$. We also give $G^p\subset D^{p,q}$ for $0<p<\infty$ and $q\geq 1$.

In the second section we denote by (A, B) the space of "multipliers" from A to B. Then we find the multipliers from D^p into G^p and from D^p into G^1 by a method of P. Ahern and M. Jevtic ([1]). We shall

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denote by $\mathscr{B}$ the space consisting of all analytic functions f for which $\sup_{0<r<1}(1-r)|f'(z)|<\infty\,(r=|z|)$. These functions are called Bloch functions. Then we show that for $1\leqq p\leqq 2$, $(\mathscr{B}, H^p)=l(\infty, 2)$ and $l(1,\infty)\subset(H^p,\mathscr{B})\subset l(2,\infty)$ by a method of J. Anderson and A. Shields ([2]). Also, we find the multipliers from D^p into $\mathscr{B}$ and from $\mathscr{B}$ into D^p.

1. Some results on the spaces $D^{p,q}$

DEFINITION 1.1. Let $f(z)=\sum_0^\infty a_k z^k$ and let β be a real number. Flett ([5]) defines the fractional integral of f of order β as

$$I^\beta f(z)=\sum(k+1)^{-\beta}a_k z^k.$$

The fractional derivative $D^\beta f$ of f of order $\beta>0$ is defined as

$$D^\beta f=I^{-\beta}f.$$

THEOREM 1.2. *$B^p\subset D^{s,q}$ for $0<s<p<1<q<\infty$.*

Proof. Let $f\in B^p$. Then $I^{(1/p)-1}f\in H^1$ (see [4], p. 77). By a theorem of Hardy and Littlewood ([3], Theorem 5.11),

$$\int_0^1(1-r)^{-1/q}M_q(r, I^{(1/p)-1}f)\ dr<\infty.$$

Hence $I^{(1/p)-1}f\in D^{1,q}$. By Theorem 7 of [1], $f\in D^{s,q}$ for all $s<p$.

COROLLARY 1.3. *$B^p\subset D^s$ for all $0<s<p<1$.*

THEOREM 1.4. *$G^p\subset D^{p,q}$ for $0<p<\infty$ and $1\leqq q<\infty$.*

Proof. If $f\in G^p$, then $\int_0^1 M_1(r,f')^p\ dr<\infty$. Hence $M_1(r,f')=0(1-r)^{-1/p}$. Since by the Cauchy formula, there is a constant C such that

$$M_\infty(r,f')\leqq C(1-r)^{-1}M_1(r,f')=0(1-r)^{-(1/p)-1}.$$

Thus

$$\int_0^1(1-r)^{-(p/q)+p}\left(\int_0^{2\pi}|f'(re^{i\theta})|^q d\theta\right)^{p/q}dr$$
$$\leqq\int_0^1(1-r)^{-(p/q)+p}M_\infty(r,f')^{p(q-1)/q}M_1(r,f')^{p/q}dr$$
$$\leqq C\int_0^1 M_1(r,f')^p\ dr<\infty.$$

COROLLARY 1. 5. (1) *$H^p \subset G^p \subset D^p$ for $0<p<1$.*

(2) *$G^p \subset H^p \subset D^p$ for $1 \leqq p < 2$.*

(3) *$G^p \subset D^p \subset H^p$ for $2 \leqq p < \infty$.*

Proof. Since $G^p = D^{p,1}$ and $D^p = D^{p,2}$, the assertions can be proved by Theorem 6 ([1]) and Theorem 1. 4.

DEFINITION 1. 6. For $1 \leqq \alpha, \beta \leqq \infty$ we denote by $l(\alpha, \beta)$ the set of those sequences $\{a_k\}$ $(k \geqq 1)$ for which

$$\{(\sum_{I_n} |a_k|^\alpha)^{1/\alpha}\}_{n=0}^{\infty} \in l^\beta \quad (\alpha < \infty)$$

and

$$\{\sup_{k \in I_n} |a_k|\}_{n=0}^{\infty} \in l^\beta \qquad (\alpha = \infty)$$

where $I_n = \{k : 2^n \leqq k < 2^{n+1}\}$. We remark that $l(p, p) = l^p$. The multipliers $(l(\alpha, \beta),\ l(\alpha', \beta'))$ are easily determined, this has been done by Kellogg [8].

The Köthe dual, denoted by A^K, is defined to be (A, l^1), the multipliers from A to l^1.

COROLLARY 1. 7. *If $f(z) = \sum_0^\infty a_k z^k \in G^p$ $(0 < p < \infty)$, then $\sum_0^\infty (k+1)^{p-2} |a_k|^p < \infty$.*

Proof. If $0 < p \leqq 2$, it is a well-known result of Holland and Twomey [7]. If $2 \leqq p < \infty$, $(G^p, l^p) = ((G^p)^{KK}, l^p)$ by Lemma 4 (iii) of [2]. By Theorem 3 of [1], $(G^p)^{KK} = \{\{\lambda_k\} : \{(k+1)^{1-(1/p)} \lambda_k\} \in l(\infty, p)\}$. Thus

$$(G^p, l^p) = \{\{\lambda_k\} : \{(k+1)^{(1/p)-1} \lambda_k\} \in l(p, \infty)\}.$$

Hence $\{(k+1)^{1-(2/p)}\} \in (G^p, l^p)$. This completes the proof.

2. Some results on multipliers

We use an approach for finding coefficient multipliers due to J. Anderson and A. Shields [2]. If A is a sequence space A^a is defined to be the set of sequences $\{\lambda_n\}$ such that $\lim_{r \to 1} \sum_0^\infty \lambda_n a_n r^n$ exists for all $\{a_n\} \in A$. A sequence space A is said to be solid if, whenever it contains $\{a_n\}$ it also contains all sequences $\{b_n\}$ with $|b_n| \leqq |a_n|$. We denote the largest solid subspace of A by $s(A)$.

If f is holomorphic in $|z|<1$, the space $A^{p,q,\alpha}$ is defined as follows:

$$A^{p,q,\alpha}=\{f:\int_0^1(1-r)^\alpha\ M_q(r,f)^p dr<\infty\}$$
$$(p>0,\ q>0,\ \alpha>-1).$$

THEOREM 2.1. *If* $1<p<\infty$, $(D^p, G^p)=\{\{\lambda_k\}:\{(k+1)^{1/2}\lambda_k\}\in l^\infty\}$.

Proof. In [1], $D^p=\{\{\lambda_k\}:\{(k+1)^{(1/2)-(1/p)}\lambda_k\}\in l(2,p)\}$. Since $G^p=(A^{p',\infty,0})^\alpha$,

$$s(G^p)=\{\{\lambda_k\}:\{(k+1)^{1/p'}\lambda_k\}\in l(2,p)\}$$

(see [1], Theorem 1, 2, 4). Here p' denotes the conjugate index. From Lemma 3 (ii) of [2] we have $(D^p, G^p)=(D^p, s(G^p))$.

Thus

$$(D^p,G^p)=\{\{\lambda_k\}:\{(k+1)^{1/2}\lambda_k\}\in(l(2,p),l(2,p))\}.$$
$$=\{\{\lambda_k\}:\{(k+1)^{1/2}\lambda_k\}\in l^\infty\}.$$

THEOREM 2.2. *If* $p\geqq 1$, $(D^p, G^1)=\{\{\lambda_k\}:\{(k+1)^{(1/p)-(1/2)}\lambda_k\}\in l(\infty,p')\}$.

Proof. In [2], $(D^p,G^1)=(D^p,s(G^1))$ and $s(G^1)=l(2,1)$.

REMARK. It is well known that the dual space of G^1 is the space $\mathscr{B}$ of the Bloch functions [2].

THEOREM 2.3. *If* $1\leqq p\leqq 2$, $(\mathscr{B}, H^p)=l(\infty,2)$.

Proof. In [2], $l(1,\infty)=s(\mathscr{B})\subset\mathscr{B}\subset l(2,\infty)$. Since $s(H^p)=l^2$ ([9], Chapter 5, Theorem 8.12) we have $(\mathscr{B}, H^p)\supset(\mathscr{B}, s(H^p))=(\mathscr{B}, l^2)=l(\infty,2)$. By Lemma 3 (ii) of [2],

$$(\mathscr{B},H^p)\subset(s(\mathscr{B}),H^p)=(l(1,\infty),H^p)$$
$$=(l(1,\infty),s(H^p))=(l(1,\infty),l(2,2))$$
$$=l(\infty,2).$$

LEMMA 2.4. $(H^1, l(1,\infty))=l(1,\infty)$.

Proof. Since $H^1\subset l(\infty,2)$ ([9], Chapter XII, Theorem 7.8),

$$(H^1,l(1,\infty))\supset(l(\infty,2),l(1,\infty))=l(1,\infty).$$

Conversely, since $G^1 \subset H^1$ and $(G^1)^{KK} = l(\infty, 1)$ we have

$$(H^1, l(1, \infty)) \subset (G^1, l(1, \infty)) = ((G^1)^{KK},\ l(1, \infty))$$
$$= (l(\infty, 1), l(1, \infty)) = l(1, \infty).$$

THEOREM 2.5. *If* $1 \leqq p \leqq 2$, *then* $l(1, \infty) \subset (H^p, \mathcal{B}) \subset l(2, \infty)$.

Proof. By Lemma 2.4 and Lemma 3 ([2]) we have

$$(H^p, \mathcal{B}) \supset (H^p, s(\mathcal{B})) = (H^p, l(1, \infty)) \supset (H^1, l(1, \infty)) = l(1, \infty)$$

and

$$(H^p, \mathcal{B}) \subset (s(H^p), \mathcal{B}) = (l^2, \mathcal{B}) = (l^2, s(\mathcal{B})) = (l^2, l(1, \infty)) = l(2, \infty).$$

REMARK. We now state some other results which can be proved by similar method given in Theorem 2.5.

(1) $l^2 \subset (H^1, G^1) \subset l(\infty, 2)$.
(2) $l(2, \infty) \subset (G^1, H^1) \subset l^\infty$.
(3) $l^2 \subset (\mathcal{B}, H^p) \subset l(\infty, 2)$ *for* $2 \leqq p < \infty$.

THEOREM 2.6. (1) $(D^p, \mathcal{B}) = \{\{\lambda_k\} : \{(k+1)^{(1/p)-(1/2)}\lambda_k\} \in l(2, \infty)\}$ *for* $p \geqq 1$.

(2) $(\mathcal{B}, D^p) = \{\{\lambda_k\} : \{(k+1)^{(1/2)-(1/p)}\lambda_k\} \in l(\infty, p)\}$ *for* $p \geqq 1$.

Proof. (1) $(D^p, \mathcal{B}) = (D^p, s(\mathcal{B})) = (D^p, l(1, \infty))$

(2) Since $D^p = (D^p)^{KK}$, $(\mathcal{B}, D^p) = (l(2, \infty), D^p)$ (see [2], p. 259).

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