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ON THE WALSH FOURIER COEFFICIENTS

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1. Introduction

Let $\{\phi_n\}$ be a system of functions orthonormal and uniformly bounded over an interval (a, b). Suppose $|\phi_n(x)| \leqq M$ for all $x \in (a, b)$ and all n.

We have the following Paley's theorems on Fourier Coefficients. (For the proofs, see [7])

Given a sequence of numbers $c_1, c_2, \cdots$ we write

$$\mathscr{B}_r[c] = \{\sum_k |c_k|^r k^{r-2}\}^{1/r}.$$

THEOREM 1.1. (i) *If $f \in L^p$, $1 < p \leqq 2$, and if $c_1, c_2, \cdots$ are the Fourier coefficients of f with respect to $\phi_1, \phi_2, \cdots$, then $\mathscr{B}_p[c]$ is finite and $\mathscr{B}_p[c] \leqq A_p M^{(2-p)/p} \|f\|_p$, where A_p is a constant depending on p.*

(ii) *If given numbers $c_1, c_2, \cdots$ satisfy the condition $\mathscr{B}_q[c] < \infty$ for some $q \geqq 2$, then there is an $f \in L^q$ having the c_n as its Fourier coefficients with respect to $\{\phi_n\}$, and such that $\|f\|_q \leqq A_q' M^{(q-2)/q} \mathscr{B}_q[c]$.*

The function f is the limit, in L^q, of $s_n = c_1\phi_1 + \cdots + c_n\phi_n$ as $n \to \infty$.

(iii) *Moreover, we may take $A_q' = A_{q'}$, where q' is the conjugate exponent of q.*

Given a sequence of complex numbers $c_k \to 0$, we denote by $\{c_k^*\}$ the sequence $|c_1|, |c_2|, \cdots$ rearranged in decreasing order. Write $\mathscr{B}_r^*[c] = \mathscr{B}_r[c^*]$.

THEOREM 1.2. (i) *Under the hypothesis of theorem 1.1 we have $\mathscr{B}_p^*[c] \leqq A_p M^{(2-p)/p} \|f\|_p$.*

(ii) *If $c_1, c_2, \cdots$ are complex numbers tending to 0 such that $\mathscr{B}_q^*[c]$ is finite, then the c_n are the Fourier coefficients, with respect to the $\{\phi_n\}$, of an $f \in L^q$ satisfying $\|f\|_q \leqq A_q' M^{(q-2)/q} \mathscr{B}_q^*[c]$.*

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In this paper we apply the above theorems to Walsh Fourier coefficients and give a necessary and sufficient condition that a certain sequence of numbers should be the Walsh Fourier coefficients of a function in some L^p.

2. Main results

We first give the Walsh versions of the theorems 1.1 and 1.2.

Consider the Walsh functions $\{w_n\}$, $n=1, 2, \cdots\cdots$, of orthonormal and uniformly bounded, $|w_n(x)|=1$, over an interval $[0, 1)$.

Given a sequence of numbers $c_1, c_2, \cdots$ we write

$$\mathscr{B}_{r,w}[c]=\{\sum_k |c_k|^r (2^{k_1+1})^{r-2}\}^{1/r},$$

where k_1 is the natural number which satisfies $2^{k_1} \leqq k < 2^{k_1+1}$ for each k and $\mathscr{B}_{r,w}{}^*[c]=\mathscr{B}_{r,w}[c^*]$.

THEOREM 2.1. (i) *If $f \in L^p(0, 1)$, $1<p\leqq 2$, and if $c_1, c_2, \cdots$ are the Walsh Fourier coefficients of f with respect to $w_1, w_2, \cdots$, then $\mathscr{B}_{p,w}[c]$ is finite and $\mathscr{B}_{p,w}[c] \leqq A_p \|f\|_p$.*

(ii) *If given numbers $c_1, c_2, \cdots$ satisfy the condition $\mathscr{B}_{q,w}[c]<\infty$ for some $q \geqq 2$, then there is an $f \in L^q$ having the c_n as its Walsh Fourier coefficients with respect to $\{w_n\}$, and such that $\|f\|_q \leqq A_q{}' \mathscr{B}_{q,w}[c]$. The function f is the limit, in L^q, of $s_n=c_1w_1+\cdots+c_nw_n$ as $n \to \infty$.*

(iii) *Moreover, we may take $A_q{}'=A_{q'}$.*

Proof. (i) Since $1<p\leqq 2$, $\mathscr{B}_{p,w}[c] \leqq \mathscr{B}_p[c]$.

By theorem 1.1 (i), with $M=1$, (i) holds.

(ii) Since $q \geqq 2$, $\mathscr{B}_q[c] \leqq \mathscr{B}_{q,w}[c]$.

By theorem 1.1 (ii), with $M=1$, (ii) holds.

THEOREM 2.2. (i) *Under the hypothesis of theorem 2.1 (i), we have $\mathscr{B}_{p,w}{}^*[c] \leqq A_p \|f\|_p$.*

(ii) *If $c_1, c_2, \cdots$ are complex numbers tending to 0 such that $\mathscr{B}_{q,w}{}^*[c]$ is finite, then the c_n are the Walsh Fourier coefficients, with respect to the $\{w_n\}$, of an $f \in L^q$ satisfying $\|f\|_q \leqq A_q{}' \mathscr{B}_{q,w}{}^*[c]$.*

Proof. (i) since $1<p\leqq 2$, $\mathscr{B}_{p,w}{}^*[c] \leqq \mathscr{B}_p{}^*[c]$.

By theorem 1.2 (i), with $M=1$, (i) holds.

(ii) Since $q \geqq 2$, $\mathscr{B}_q{}^*[c] \leqq \mathscr{B}_{q,w}{}^*[c]$.

By theorem 1.2 (ii), with $M=1$, (ii) holds.

We now give a necessary and sufficient condition that a certain sequence of numbers should be the Walsh Fourier coefficients of a function in some L^p.

Given a sequence $c_0, c_1, c_2 \cdots$ tending to 0, let $c_0^* \geqq c_1^* \geqq c_2^* \geqq \cdots$ be the sequence $|c_0|$, $|c_1|$, $|c_2|$, $\cdots$ rearranged in descending order of magnitude.

THEOREM 2. 3. (i) *A necessary and sufficient condition that numbers $c_n \to 0$ should be, for every variation of their arrangement, the Walsh Fourier coefficients of a function $f \in L^q$, $q \geqq 2$, is that $\mathscr{B}_{q,w}^*[c] < \infty$. If the condition is satisfied, then for every such f,*

$$\|f\|_q \leqq A_q' \mathscr{B}_{q,w}^*[c] \cdots\cdots (*)$$

(ii) *A necessary and sufficient condition that the c_n should be, for some variation of their arrangement, the Walsh Fourier coefficients of an $f \in L^p$, $1 < p \leqq 2$, is that $\mathscr{B}_{p,w}^*[c] < \infty$. Moreover, we have, for every such f,*

$$\mathscr{B}_{p,w}^*[c] \leqq A_p \|f\|_p \cdots\cdots (**)$$

The proof is based on the following lemmas.

LEMMA 2. 4. ([7]) *Suppose f is a non-negative function defined for $x \geqq 0$. Let $r > 1$ and $s < r-1$. If $f^r(x) x^s$ is integrable over $(0, \infty)$, so is $\{x^{-1}F(x)\}^r x^s$, where $F(x) = \int_0^x f \, dt$.*

LEMMA 2. 5. *If $a_0 \geqq a_1 \geqq a_2 \geqq \cdots \to 0$, a necessary and sufficient condition that the function $g(x) = \Sigma a_n w_n(x)$ should belong to L^r, $r > 1$, is that the sum $S_r = \Sigma a_n^r (2^{n_1+1})^{r-2}$, where $2^{n_1} \leqq n < 2^{n_1+1}$, should be finite.*

Proof. Let $G(x)$ and $H(x)$ denote, respectively, the integrals of g and $|g|$ over the interval $(0, x)$. Let $A_n = a_1 + a_2 + \cdots + a_n$. By B we shall mean a constant depending at most on r, but not necessarily always the same.

If $g \in L$, in particular if $g \in L^r$, the series defining g is $S[g]$. So

$$\begin{aligned} G(1/2^n) &= \int_0^{1/2^n} g(t)\, dt \\ &= \int_0^{1/2^n} [\sum_0^\infty a_k w_k(t)]\, dt \\ &= \sum_{k=0}^\infty a_k \int_0^{1/2^n} w_k(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{2^n-1} a_k\right)\cdot 1/2^n \\
&\geqq 2^n\cdot a_{2^n-1}\cdot 1/2^n \\
&= a_{2^n-1} \\
&\geqq a_{2^n},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{\infty} a_n^r (2^{n_1+1})^{r-2} &= [a_2^r + a_3^r](2^2)^{r-2} + [a_4^r + a_5^r + a_6^r + a_7^r](2^3)^{r-2} + \cdots \\
&\leqq 2\cdot a_2^r (2^2)^{r-2} + 2^2\cdot a_{2^2}^r (2^3)^{r-2} + \cdots \\
&= \sum_{n=1}^{\infty} a_{2^n}^r\, 2^n (2^{n+1})^{r-2} \\
&\leqq \sum_{n=1}^{\infty} G^r(1/2^n)(2^n)^{r-1}\cdot 2^{r-2} \\
&\leqq \sum_{n=1}^{\infty} \left(\int_0^{1/2^n} |g(t)|\,dt\right)^r (2^n)^{r-1}\cdot 2^{r-2} \\
&= \sum_{n=1}^{\infty} H^r(1/2^n)\left(\frac{1}{1/2^n}\right)^{r-1} 2^{r-2} \\
&\leqq \sum_{n=1}^{\infty} \int_{1/2^n}^{1/2^{n-1}} \left(\frac{H(x)}{x}\right)^r (2^n)^{-1} 2^{r-2} \\
&\leqq \sum_{n=1}^{\infty} \int_{1/2^n}^{1/2^{n-1}} \left(\frac{H(x)}{x}\right)^r 2^{r-2} \\
&= B\int_0^1 \left(\frac{H(x)}{x}\right)^r \\
&\leqq B\left(\frac{r}{r-1}\right)^r \int_0^1 |g|^r dx \\
&= B\int_0^1 |g|^r dx,
\end{aligned}
$$

by lemma 2.4.

This establishes the necessity of the condition in 2.5.

To show that the condition is sufficient we observe that

$$
\begin{aligned}
|g(x)| &= \left|\sum_{\nu=0}^{\infty} a_\nu w_\nu(x)\right| \\
&\leqq \left|\sum_{\nu=0}^{2^n-1} a_\nu w_\nu(x)\right| + \left|\sum_{\nu=2^n}^{\infty} a_\nu w_\nu(x)\right|
\end{aligned}
$$

$$\leqq \sum_{\nu=0}^{2^n-1} a_\nu + \Big| \sum_{\nu=2^n}^{\infty} (a_\nu - a_{\nu+1}) D_{\nu+1}(x) - a_{2^n} D_{2^n} \Big|$$

$$\leqq \sum_{\nu=0}^{2^n-1} a_\nu + \Big| \sum_{\nu=2^n}^{\infty} (a_\nu - a_{\nu+1}) D_{\nu+1}(x) + a_{2^n} D_{2^n} \Big|$$

$$\leqq \sum_{\nu=0}^{2^n-1} a_\nu + \sum_{\nu=2^n}^{\infty} (a_\nu - a_{\nu+1}) |D_{\nu+1}(x)| + a_{2^n} |D_{2^n}|$$

$$\leqq \sum_{\nu=0}^{2^n-1} a_\nu + a_{2^n} \frac{2^2}{x}.$$

If follows that $|g(x)| \leqq BA_{2^n}$ for $\frac{1}{2^{n+1}} \leqq x \leqq \frac{1}{2^n}$.

Hence $\int_0^1 |g|^r dx = \sum_0^\infty \int_{1/2^{n+1}}^{1/2^n} |g|^r dx < B \Sigma A_{2^n}^r \frac{1}{2^{n+1}} < B \Sigma A_{2^n}^r \frac{1}{n^2}$, and it remains to show that the last series converges if $S_r < \infty$. $S_r < \infty$ implies $\Sigma a_n^r n^{r-2} < \infty$.

Let $a(x)$ denote the function equal to a_n for $n-1 \leqq x < n$ $(n=1, 2, \cdots)$, and $A(x)$ the integral of $a(t)$ over $(0, x)$. The inequality $\Sigma a_n^r n^{r-2} < \infty$ implies that $a^r(x) x^{r-2}$ is integrable over $(0, \infty)$. So (by lemma 2.4 with $s=r-2$) is the function $\{x^{-1} A(x)\}^r x^{r-2} = A^r(x) x^{-2}$.
The integrability of the latter function is equivalent to the convergence of $\Sigma A_n^r n^{-2}$. Hence $\Sigma A_{2^n}^r n^{-2}$ converges and lemma 2.5 follows.

We are now in a position to prove theorem 2.3.

(i) That the condition of (i) is sufficient follows from (ii) of theorem 2.2, whence also we can deduce the inequality (*).

To prove the necessity of the condition, consider the series $\Sigma c_n^* w_n(x)$. Since $\Sigma c_n^* w_n(x)$ belongs to L^q, by lemma 2.5 we see that

$$\mathscr{B}_{q,w}^*[c] = \{\Sigma (c_n^*)^r (2^{n_1+1})^{r-2}\}^{1/r} < \infty, \text{ where } 2^{n_1} \leqq n < 2^{n_1+1}.$$

(ii) By (i) of theorem 2.2, we see that the condition of theorem 2.3 (ii) is necessary. Moreover, we can deduce the inequality (**).

To prove the sufficiency of the condition, consider the series $\Sigma c_n^* w_n(x)$. Since $\mathscr{B}_{p,w}^*[c] < \infty$, by lemma 2.5 $\Sigma c_n^* w_n(x)$ belongs to L^p.

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