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ON THE PERFECT IDEALS

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In this paper we will study the type of the perfect ideal in the noetherian local ring R with the maximal ideal m. The terminology and theorems are most like [1] and [2].

Recall the grade of I is the length of the maximal R-sequence contained in I. (Bourbaki calls this the I-depth of R.) In case R is Cohen-Macaulay (shortly C-M), grade I=height of I.

We call I a perfect ideal when grade I=p.d. (R/I), the projective dimension of R/I. In fact, if R is C-M and I is a perfect ideal, then R/I is C-M by the Anslander-Buchsbaum formula depth R/I+p.d. (R/I)=depth R.

Now, if I is a perfect ideal of grade g, then $\mathrm{Ext}_R{}^g(R/I, R/m)\neq 0$ since p.d. $(R/I)=g$. Hence we can have the following definition.

Definition 1. Let I be a perfect ideal of grade g. Then the type of I is defined as $\dim_{R/m} \mathrm{Ext}_R{}^g(R/I, R/m)$.

First, we have the following short proposition.

Proposition 2. *Let I be a perfect ideal of grade g and*

$$0\to F_g\xrightarrow{d_g}F_{g-1}\to\cdots\to F_1\xrightarrow{d_1}R\to R/I\to 0$$

be the minimal free resolution of R/I. Then the type of I is equal to the rank of F_g.

Proof. $\mathrm{Ext}_R{}^g(R/I, R/m)=\mathrm{Hom}_R(F_g, R/m)/\mathrm{Image}(\mathrm{Hom}(d_g, l_{R/m}))$. But since $d_g(F_g)\subseteq mF_{g-1}$, $\mathrm{Hom}(d_g, l_{R/m}):\mathrm{Hom}_R(F_{g-1}, R/m)\to\mathrm{Hom}_R(F_g, R/m)$ is trivial. Hence $\mathrm{Ext}_R{}^g(R/I, R/m)=\mathrm{Hom}_R(F_g, R/m)$, and the type of I is the rank of F_g.

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Note that the rank of F_1 is the minimal number of generators of I, noted by $\mu(I)$.

When I is a perfect ideal, I is called Gorenstein if it has type 1.

An ideal I of grade g is complete intersection if it can be generated by g elements. In this case $I=(a_1, \cdots, a_g)$ where $a_1, \cdots, a_g$ is R-regular sequence. Then the Koszul complex $K(a_1, \cdots, a_n, R)$ is a free resolution of R/I, hence the type of I is 1, which means I is Gorenstein.

When I is a perfect ideal of grade 2, we can have the following theorem.

THEOREM 3. *Let R be a regular local ring and $I \subseteq R$ be a perfect ideal of grade 2 with $\mu(I)=n$. Then type of $I=n-1$.*

Proof. Since $\mu(I)=n$, we get the following minimal resolution of R/I.

$$0 \to F_2 \to R^n \to R \to R/I \to 0.$$

Hence

$$0 \to F_2 \to R^n \to I \to 0, \text{ exact.}$$

By tensoring with $K \equiv$ quotient field of R, we have

$$0 \to F_2 \otimes_R K \to K^n \to I \otimes_R K \to 0.$$

But $I \otimes_R K = K$, hence $F_2 \otimes_R K \simeq K^{n-1}$ so that $F_2 \simeq R^{n-1}$.

From Theorem 3, we have the following Serre's well-known Theorem.

COROLLARY 4. *Let R be a regular local ring and I be a perfect ideal of grade 2. Then I is complete intersection iff I is Gorenstein.*

Perfect ideal of grade g is almost complete intersection if it can be generated by $g+1$ elements.

COROLLARY 5. *With the same conditions in Corollary 4, almost complete intersection ideal is not Gorenstein.*

As a matter of fact, the almost complete intersection ideal I in Corollary 5 can be generated by minors of a 2×3 matrix. For R/I is C-M, we get the following free resolution

$$0 \to R^2 \xrightarrow{A} R^3 \to R \to R/I \to 0$$
$$R^3 \xrightarrow{\phi} I \to 0$$

Let $I=(a_1, a_2, a_3)$ and $\phi(x, y, z)=xa_1+ya_2+za_3$,

$$A=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\end{pmatrix}$$

Then

$$\alpha_{11}a_1+\alpha_{12}a_2+\alpha_{13}a_3=0$$
$$\alpha_{21}a_1+\alpha_{22}a_2+\alpha_{23}a_3=0$$

By [3], Theorem 3. 3 it follows that

$$a_1=\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{22},$$
$$a_2=-\alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21},$$
$$a_3=\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}.$$

As shown above, when grade $I=2$, it's quite easy to calculate the minimal free resolution. But generally it is not easy to handle the minimal free resolution even in the case grade $I=3$. Hence we'll take somewhat special condition.

THEOREM 6. *Let R be a regular local ring and I be a perfect ideal of grade 3 with $I=(I_0, x)$. Here I is a perfect ideal of grade 2 and x is not a zero divisor of R/I_0. Then type of $I=n-2$, where $n=\mu(I)$.*

Proof. By Theorem 3, we have type of $I_0=n-2$. Hence we have

$$0 \to R^{n-2} \to R^{n-1} \xrightarrow{\phi} R \to R/I_0 \to 0, \text{ exact.}$$

Let $I=(y_1, \cdots, y_{n-1})$ and $\phi(e_j)=y_j$ $j=1, \cdots, n-1$. Here $e_1, \cdots, e_{n-1}$ denote the standard basis of R^{n-1}. Define $\phi' : R^n \to R$ by $\phi'|_{R^{n-1}}=\phi$ and $\phi'(e_n)=x$. Then Im $\phi'=I$. If $\widetilde{K}$=Kernel of ϕ' then $\widetilde{K}$ is generated by Ker ϕ in R^{n-1} and the elements $xe_j-y_je_n$. Therefore we can easily check that $\widetilde{K}$ is minimally generated by $2n-3$ elements. Now we have the following exact sequences.

$$
\begin{array}{ccccccccccccc}
0 & \to & F_3 & \to & R^{2n-3} & & \longrightarrow & & R^n & \xrightarrow{\phi'} & R \to R/I \to 0 \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & \widetilde{K} & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & &
\end{array}
$$

From this we have $0 \to F_3 \to R^{2n-3} \to R^n \to I \to 0$, exact. By tensoring with $K \equiv$ quotient field of R, we have $0 \to F_3 \otimes K \to K^{2n-3} \to K^n \to K \to 0$, exact. Therefore $F_3 \otimes K \simeq K^{n-2}$, so that $F_3 \simeq R^{n-2}$ as wanted.

References

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