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WREATH PRODUCT OF TOPOLOGICAL INVERSE SEMIGROUPS

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1. Introduction

Let S be a topological semigroup which is algebraically an inverse semigroup. Then the inversion function $x \to x^{-1}$ on S is aiways one-to-one and onto. However, it may or may not be continuous. If it is continuous, then S will be called a topological inverse semigroup. This concept was introduced by Eberhart, C. and Selden, J. [3]. And they established several properties of such objects, and investigated the structure of the closure of a bicyclic semigroup considered as subsemigroup of a locally compact topological inverse semigroup. Selden, A.A. [6] investigated the structure of the closure of a bisimple ω-semigroup considered as a subsemigroup of a locally compact topological inverse semigroup. In recent years, Koray, S. [5] gave a characterization of certain subclass of bisimple locally compact topological inverse semigroups whose maximal subgroups are compact and whose set of idempotents is isomorphic to an interval of the real line.

In this paper, we first discuss a topological inverse semigroup of continuous functions from a locally compact space into a topological inverse semigroup. And we establish the wreath product of topological inverse semigroups as one of the semidirect products of topological semigroups. Many properties concerned with the wreath product of algebraic semigroups are well known in [7], [8] and related papers.

2. Preliminaries

We begin with some definitions and some results relative to topological semigroups for the sequel.

DEFINITION 2.1. A semigroup is a nonempty set S together with an associative multiplication $(x, y) \to xy$ from $S \times S$ into S. The associative condition on S states that $(xy)z = x(yz)$ for all $x, y, z \in S$.

An element e of a semigroup S is called an idempotent if $e^2=e$. $E(S)$ will denote the set of all idempotents of S.

DEFINITION 2. 2. A topological semigroup is a Hausdorff space S together with a continuous associative multiplication.

The condition that multiplication on S is continuous is equivalent to the condition that for each $x, y \in S$ and each open set W in S with $xy \in W$, there exist open sets U and V in S such that $x \in U$, $y \in V$ and $UV \subset W$.

Throughout, all topological spaces will be assumed Hausdorff. If A is a subset of a space S, then $\bar{A}$ will denote the closure of A in S.

DEFINITION 2. 3. A semigroup S is (algebraically) an inverse semigroup if each element x of S has a unique inverse; that is, there is a unique element x^{-1} of S such that $xx^{-1}x=x$ and $x^{-1}xx^{-1}=x^{-1}$.

Note that if S is an inverse semigroup then xx^{-1} and $x^{-1}x$ are idempotents of S for each $x \in S$.

DEFINITION 2. 4. A topological inverse semigroup is a topological semigroup S which is an inverse semigroup and the inversion function $x \rightarrow x^{-1}$ on S is continuous.

For other definition and terminology, we refer to [1], [2] and [4].

3. Topological inverse semigroup of continuous functions

If X and Y are Hausdorff spaces, then $C(X, Y)$ denotes the set of all continuous functions from X into Y. For $A \subset X$ and $B \subset Y$, we use $N(A, B)$ to denote the set $\{f \in C(X, Y) \mid f(A) \subset B\}$. The topology on $C(X, Y)$ having the collection of all $N(K, U)$ such that K is compact subset of X and U is an open subset of Y as subbase is called the compact-open topology on $C(X, Y)$.

For Hausdorff spaces X and Y, we will be assumed for the remainder of this section that $C(X, Y)$ is assigned the compact-open topology.

THEOREM 3. 1. *Let S be a locally compact space and let T be a topological semigroup. Then $C(S, T)$ with the pointwise multiplication is a topological semigroup.*

Proof. It is well known that $C(S, T)$ is a Hausdorff space. Let T^S denote the set of all functions from S into T. Then T^S is a semigroup under the pointwise multiplication. Since $C(S, T)$ is a subalgebra of the algebra T^S, $C(S, T)$ is a semigroup. To prove the multiplication on $C(S, T)$ is continuous: Let K be a compact subset of S, W an open subset of T. Suppose $f, g\in C(S, T)$ such that $fg\in N(K, W)$. Then $(fg)(K)\subset W$. Hence $(fg)(x)=f(x)g(x)\in W$ for each $x\in K$. By continuity of multiplication on T, for each $x\in K$, there exist open sets U_x and V_x in T such that $f(x)\in U_x$, $g(x)\in V_x$ and $U_xV_x\subset W$. Since S is locally compact and f and g are continuous, for each $x\in K$, there exist open sets A_x and C_x in S and $\bar{A}_x$ a compact set in S such that $x\in A_x\subset\bar{A}_x\subset C_x$, $f(C_x)\subset U_x$ and $g(C_x)\subset V_x$. Then $\{A_x | x\in K\}$ is an open cover of K. Since K is compact, there exists a finite cover $A_1, \cdots, A_n$ of A_x's and corresponding $\bar{A}_1, \cdots, \bar{A}_n$ of A_x's. Then $\bar{A}_1, \cdots, \bar{A}_n$ is a cover of K by compact sets. Let $U_1, \cdots, U_n$ and $V_1, \cdots, V_n$ be the corresponding collections of U_x's and V_x's respectively. Then for each $1\leq j\leq n$, we have $f(\bar{A}_j)\subset U_j$, $g(\bar{A}_j)\subset V_j$ and $U_jV_j\subset W$. Hence $f\in\bigcap_{j=1}^{n} N(\bar{A}_j, U_j)$ and $g\in\bigcap_{j=1}^{n} N(\bar{A}_j, V_j)$. Suppose $h\in\bigcap_{j=1}^{n} N(\bar{A}_j, U_j)$ and $k\in\bigcap_{j=j}^{n} N(\bar{A}_j, V_j)$and fix $x\in K$. Then $x\in\bar{A}_i$ for some $1\leq i\leq n$. So $(hk)(x)=h(x)k(x)\in h(\bar{A}_i)k(\bar{A}_i)\subset U_iV_i\subset W$. It follows that $hk\in N(K, W)$. Hence $(\bigcap_{j=1}^{n} N(\bar{A}_j, U_j))(\bigcap_{j=1}^{n} N(\bar{A}_j, V_j))\subset N(K, W)$, and hence the multiplication on $C(S, T)$ is continuous. Therefore, $C(S, T)$ is a topological semigroup.

COROLLARY 3.2. *Let S be a locally compact space and let T be an abelian semigroup. Then $C(S, T)$ with the pointwise multiplication is an abelian topological semigroup.*

THEOREM 3.3. *Let S be a locally compact space and let T be a topological inverse semigroup. Then $C(S, T)$ with the pointwise multiplication is a topological inverse semigroup.*

Proof. In view of Theorem 3.1., $C(S, T)$ is a topological semigroup. Algebraically inverse semigroups are productive and hence the power T^S is an inverse semigroup and $C(S, T)$ is also closed under the inversion function on T^S. Thus $C(S, T)$ is an inverse semigroup. Let $\phi : T\rightarrow T$ be the inversion function. For each $f\in C(S, T)$, let $\bar{f}=\phi\circ f$, then $\bar{f}$ is the inverse of f. Now, let $\rho : C(S, T)\rightarrow C(S, T)$ be the inversion function. Then $\rho(f)=\bar{f}=\phi\circ f$ for each $f\in C(S, T)$. Let K be a compact subset of S, W an open subset of T, $f\in C(S, T)$, and $\bar{f}=\rho(f)\in N(K, W)$. Then $(\phi\circ f)(K)=\bar{f}(K)=\rho(f)(K)\subset W$. Hence $f(K)\subset\phi^{-1}(W)$, and hence $f\in N(K, \phi^{-1}(W))$, where $\phi^{-1}(W)$ is an open subset of T because the inversion function ϕ is continuous. If $g\in N(K, \phi^{-1}(W))$, then $g(K)\subset\phi^{-1}(W)$. So $\rho(g)(K)=\bar{g}(K)=(\phi\circ g)(K)\subset W$, and so $\rho(g)\in N(K, W)$. Thus $N(K, \phi^{-1}(W))\subset N(K, W)$. Hence ρ is continuous. Therefore, $C(S, T)$ is a topological inverse semigroup.

COROLLARY 3. 4. *Let S be a locally compact space and let T be an abelian topological inverse semigroup. Then $C(S, T)$ is an abelian topological inverse semigroup.*

4. Wreath product of topological inverse semigroups

If S is a [topological] semigroup, then we use End(S) to denote the set of [continuous] endomorphisms of S.

Note that if S is a [locally compact] semigroup then End(S) [with the relative topology of $C(S, S)$] is a [topological] semigroup under the composition of [continuous] homomorphisms ([1]).

DEFINITION 4. 1. Let S be a [locally compact] semigroup and let T be a [topological] semigroup. If there exists a [continuous] homomorphism $\phi: T\to\mathrm{End}(S)$, then we define the semidirect product $S\times_\phi T$ of S and T to be $S\times T$ [with the product topology] together with multiplication $((s_1, t_1),\ (s_2, t_2))\to(s_1\phi(t_1)(s_2),\ t_1t_2)$.

LEMMA 4. 2. *Let S be a [locally compact] semigroup, T a [topological] semigroup, and $\phi: T\to\mathrm{End}(S)$ a [continuous] homomorphism. Then $S\times_\phi T$ is a [topological] semigroup. (see [1], page 67)*

DEFINITION 4. 3. Let S and T be semigroups and let S^T be the set of all functions from T into S. The wreath product $S\wr T$ of S and T is the set $S^T\times T$ with multiplication defined by $((f, a),\ (g, b))\to(fg_a, ab)$ for all $f, g\in S^T$ and $a, b\in T$, where $(fg_a)(x)=f(x)g(xa)$ for all $x\in T$.

REMARK. Suppose S and T are semigroups. Then the set S^T of all functions from T into S is a semigroup under the pointwise multiplication. Define $\phi: T\to\mathrm{End}(S^T)$ by $\phi(t)(f)=f\circ\rho_t$, where ρ_t is a right translation by $t\in T$. Then ϕ is a homomorphism. Hence the wreath product $S\wr T$

of S and T is $S^T \times_\phi T$, and hence $S \wr T = S^T \times T$ with multiplication given by $((f, a), (g, b)) \rightarrow (fg \circ \rho_a, ab)$

In view of Lemma 4. 2. and Remark, the following theorems are easily obtained.

THEOREM 4. 4. *Let S and T be semigroups. Then the wreath product $S \wr T$ of S and T is a semigroup.*

THEOREM 4. 5. *Let S be a topological semigroup and let T be a locally compact topological semigroup. Suppose that the semigroup S^T of all continuous functions from T into S is locally compact and suppose $\phi: T \rightarrow$ End(S^T) given by $\phi(a)(f) = f \circ \rho_a$ is continuous. Then the wreath product $S \wr T$ of S and T is a topological semigroup.*

THEOREM 4. 6. *Let S and T be inverse semigroups. If $f(xe) = f(x)$ for all $e \in E(T)$, $x \in T$ and $f \in S^T$, then the wreath product $S \wr T$ of S and T is an inverse semigroup.*

Proof. In view of Lemma 4. 2, $S \wr T$ is a semigroup. Let $(f, a) \in S \wr T = S^T \times T$. Define $(f, a)^{-1} = (g, a^{-1})$ such that $g(x) = (f(xa^{-1}))^{-1}$ for all $x \in T$. Then $(f, a)(f, a)^{-1}(f, a) = (f, a)(g, a^{-1})(f, a) = (fg_a f_{aa^{-1}}, aa^{-1}a) = (fg_a f_{aa^{-1}}, a)$, where $(fg_a f_{aa^{-1}})(x) = f(x)g(xa)f(xaa^{-1}) = f(x)f(xaa^{-1})^{-1} f(x) = f(x)f(x)^{-1}f(x) = f(x)$ for all $x \in T$. Hence $(f, a)(f, a)^{-1}(f, a) = (f, a)$. Similarly, we have $(f, a)^{-1}(f, a)(f, a)^{-1} = (f, a)^{-1}$. To prove the uniqueness of inverse, we assume that $(f, a)(h, b)(f, a) = (f, a)$ and $(h, b)(f, a)(h, b) = (h, b)$ for some $(h, b) \in S \wr T$. Then $(fh_a f_{ab}, aba) = (f, a)$ and $(hf_b h_{ba}, bab) = (h, b)$. These imply that $f(x)h(xa)f((xab) = f(x)$, $h(x)f(xb)h(xba) = h(x)$ for all $x \in T$ and $b = a^{-1}$. These imply that $f(x)h(xa)f(x) = f(x)$ and $h(x)f(xa^{-1})h(x) = h(x)$ for all $x \in T$. For $h(x)f(xa^{-1})h(x) = h(x)$ for all $x \in T$, $h(xa)f(xaa^{-1})h(xa) = h(xa)$ for all $x \in T$. So $h(xa)f(x)h(xa) = h(xa)$ for all $x \in T$. It follows that $h(xa) = f(x)^{-1}$ for all $x \in T$. This implies that $h(x) = h(xa^{-1}a) = f(xa^{-1})^{-1} = g(x)$ for all $x \in T$. Hence $h = g$, and hence $(h, b) = (g, a^{-1}) = (f, a)^{-1}$. Therefore, $S \wr T$ is an inverse semigroup.

THEOREM 4. 7. *Let S be a topological semigroup and let T be a locally compact topological innverse semigroup. Suppose that the semigroup S^T of all continuous functions from T into S is locally compact and suppose $\phi: T \rightarrow$ End(S^T) given by $\phi(a)(f) = f \circ \rho_a$ is continuous. If $f(xe) = f(x)$*

for all $e \in E(T)$, $x \in T$ *and* $f \in S^T$, *then the wreath product* $S \wr T$ *of* S *and* T *is a topological inverse semigroup.*

Proof. In view of Theorem 4.5, $S \wr T$ is a topological semigroup. In view of Theorem 4.6, $S \wr T$ is an inverse semigroup. We need to show that the inversion function $S \wr T = S^T \times T$ is continuous; To prove this, we adopt the following notations;

(i) Inv_{S^T} and Inv_T are inversion functions on S^T and T respectively,

(ii) $\Pi_1 : S^T \times T \to S^T$ is the first projection, and

(iii) $\Pi_2 : S^T \times T \to T$ is the second projection.

Then inversion function on $S \wr T$ is $(\mathrm{Inv}_{S^T} \circ \phi(a^{-1}) \circ \Pi_1) \times (\mathrm{Inv}_T \circ \Pi_2)$. Hence it is continuous. Therefore, $S \wr T$ is a topological inverse semigroup.

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