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ENVELOPING SEMIGROUPS AND EQUICONTINUITY OF TRANSFORMATION GROUPS

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0. Introduction

The enveloping semigroup of a transformation group (X, T, π) has been introduced by R. Ellis [1] as the closure of the set $\pi^T=\{\pi^t \mid t\in T\}$ for the p.c. topology on X^X, having a semigroup structure. In the present paper, we introduce the u.c. enveloping semigroup of (X, T, π) as the closure of π^T for the u.c. topology on X^X having a semigroup structure. The purpose of this paper is to find some properties on the u.c. enveloping semigroups and the equicontinuity of transformation groups.

Throughout this paper we assume that X is the uniform space with a uniformity $\mathscr{U}$ and the uniform topology. For every $U\in\mathscr{U}$ and $x\in X$, $U[x]$ denotes the set of all y such that (x, y) belongs to U and X^X denotes the set of all functions of X into itself.

1. Preliminaries

A subspace S of X is said to be *precompact* if for each U in $\mathscr{U}$ there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of S such that $S\subset\bigcup_{i=1}^{n} U[x_i]$. Every subspace of a precompact uniform space is also precompact. X is compact iff it is precompact and complete.

A subset F of X^X is said to be *equicontinuous* if for each U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ such that $(x, y)\in V$ implies $(f(x), f(y))\in U$ for all $f\in F$.

For each $\mathscr{U}$ in U and x in X, U_x denotes the set of all (f, g) in $X^X\times X^X$ such that $(f(x), g(x))\in U$ and U_X the set of all (f, g) in $X^X\times X^X$ such that $(f(y), g(y))\in U$ for all $y\in X$. Then the family $\{U_X \mid U\in\mathscr{U}\}$ is a base for a uniformity for X^X, and the family $\{U_x \mid U\in$

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$\mathscr{U}$, $x\in X\}$ is a subbase for a uniformity for X^X.

The uniformity with the base $\{U_X | U\in\mathscr{U}\}$ is called the *uniformity of uniform convergence* for X^X, denoted by $\mathscr{U}_u$, and the uniformity with the subbase $\{U_x | U\in\mathscr{U},\ x\in X\}$ is called the *uniformity of pointwise convergence* for X^X, denoted by $\mathscr{U}_p$. The uniform topology of $\mathscr{U}_u$ is called the *topology of uniform convergence* or simply the *u.c. topology* on X^X, and the uniform topology of $\mathscr{U}_p$ is called the *topology of pointwise convergence* or simply the *p.c. topology* on X^X.

The u.c. topology is finer than the p.c.topology since $U_X[f]$ is contained in $U_x[f]$ for all $(U, x, f)\in\mathscr{U}\times X\times X^X$. If X is complete, then so is the uniform space $(X^X, \mathscr{U}_u)$. We also can show that if X is precompact then a subspace F of $(X^X, \mathscr{U}_u)$ is precompact iff for each U in $\mathscr{U}$ there exists a finite partition $\mathscr{A}$ of X such that $(f, A)\in F\times\mathscr{A}$ implies $f(A)\times f(A)\subset U$. Thus we obtain the followings.

THEOREM 1. 1. *Let X be compact and let F be a subspace of* $(X^X, \mathscr{U}_u)$ *consisting of continuous functions. Then F is precompact iff it is equicontinuous.*

Proof. Given any member U of $\mathscr{U}$, there exists a symmetric member V of $\mathscr{U}$ with $V^3\subset U$ and there exists a finite subset G of F such that $F\subset\bigcup\{V_X[g] | g\in G\}$. For each f in F there exists a g_f in G such that $V_X[g_f]$ contains f. But the function g_f is uniformly continuous since X is compact. Hence for every g_f in G there exists a W_{g_f} in $\mathscr{U}$ such that $(x, y)\in W_{g_f}$ implies $(g_f(x), g_f(y))\in V$.

Now we set $W=\bigcap\{W_{g_f} | f\in F\}$. Then it is member of $\mathscr{U}$ since the set G is finite. Moreover for each f in F and (x, y) in W,

$$(f(x), f(y))=(f(x), g_f(x))\circ(g_f(x), g_f(y))\circ(g_f(y), f(y))\in V^3\subset U.$$

Conversely, suppose that F is equicontinuous and U is any member of $\mathscr{U}$. Then there exists a V in $\mathscr{U}$ such that $(f(x), f(y))\in U$ for all $f\in F$ and $(x, y)\in V$. Since X is precompact there exists a finite partition $\mathscr{A}$ of X such that $A\times A\subset V$ for all $A\in\mathscr{A}$. Thus $f(A)\times f(A)\subset U$ for all (f, A) in $F\times\mathscr{A}$. Therefore F is precompact.

THEOREM 1. 2. *If F is an equicontinuous subset of* X^X *equipped with the relative topology for the u.c. topology, then the map* $\mu: (x, f)\mapsto f(x)$ *of* $X\times F$ *into X is continuous.*

Proof. Let (x, f) be a point of $X\times F$ and let U and V be any members of $\mathscr{U}$ with $V^2 \subset U$. Since F is equicontinuous there exists a W in $\mathscr{U}$ such that $(g(y),\ g(z)) \in V$ for all $g \in F$ and $(y, z) \in W$. Then $W[x] \times (V_X[f] \cap F)$ is a neighborhood of (x, f) in $X \times F$ such that $U[f(x)]$ contains $\mu(W[x] \times (V_X[f] \cap F))$.

In fact, if $(u, p) \in W[x] \times (V_X[f] \cap F)$ then $(p(x), p(u)) \in V$ and $(f(x), p(x)) \in V$, so that

$$\begin{aligned}(f(x),\ \mu(u,p)) &= (f(x),\ p(u)) \\ &= (f(x),\ p(x)) \circ (p(x),\ p(u)) \in V^2 \subset U.\end{aligned}$$

2. The Enveloping Semigroups

In the remainder of this paper, we assume that X is compact Hausdorff.

The set X^X is provided with the semigroup structure such that for any f and g in X^X, $fg : X \to X$ is defined by $(fg)(x) = g(f(x))$ for all x in X. Then for each g in X^X the left translation L_g on the semigroup X^X is continuous for the u.c. topology. If a function g in X^X is continuous then the right translation R_g on the semigroup X^X is continuous for the u.c. topology.

We also can show that for a transformation group (X, T, π) the closure of π^T for the u.c. topology on X^X is a subsemigroup of X^X.

DEFINITION 2.1. The closure of π^T for the u.c. topology on X^X with the semigroup structure will be called the *u.c. enveloping semigroup* of the transformation group (X, T, π) and denoted by $E_u(X)$ or E_u. On the other hand, the *enveloping semigroup* defined by R. Ellis is denoted by $E(X)$ or E.

REMARK 2.2. For a transformation group (X, T, π), the u.c. enveloping semigroup $E_u(X)$ is a sub-semigroup of the enveloping semigroup $E(X)$.

THEOREM 2.3. *Every function in $E_u(X)$ is continuous. Hence, every right or left translation on $E_u(X)$ is continuous for the relative topology for the u.c. topology on X^X.*

Proof. Let p be any function in $E_u(X)$ and x any point of X. Let U be a member of $\mathscr{U}$ with $V^3 \subset U$ for some symmetric member V of $\mathscr{U}$. Then there exists a t in T such that $\pi^t \in V_X[p]$.

Moreover, since π^t is continuous, there exists a W in $\mathscr{U}$ such that $\pi^t(W[x]) \subset V[\pi^t(x)]$. Since $z \in W[x]$ implies $(\pi^t(x), \pi^t(z)) \in V$,

$(p(x), p(z))=(p(x), \pi^t(x))\circ(\pi^t(x), \pi^t(z))\circ(\pi^t(z), p(z))\in V^3\subset U.$
This means that $p(W[x])\subset U[p(x)]$ and p is continuous.

REMARK 2.4. In general, it is not true that for each q in $E(X)$ the right translation R_q on $E(X)$ is continuous. In particular, if (X, T, π) is almost periodic then it is true [1].

THEOREM 2.5. *The map $\nu: (p, t) \mapsto p\pi^t$ of $E_u\times T$ into E_u defines an action on E_u, where E_u has the relative topology for the u.c. topology and T has the discrete topology.*

Proof. It is sufficient to show that ν is continuous. Let a net $\{(p_\alpha, t_\alpha)\}$ converge to (p, t) in $E_u\times T$. For each α, the right translation $p\mapsto p\pi^{t_\alpha}$ on E_u is continuous, so $\{p_\beta\pi^{t_\alpha}\}$ converges to $p\pi^{t_\alpha}$. But the map $t\mapsto\pi^t$ of T into E_u is continuous since T is discrete. Hence $\{\pi^{t_\alpha}\}$ converges to π^t in E_u. Moreover, $\{p\pi^{t_\alpha}\}$ converges to $p\pi^t$ in E_u since the left translation L_p on E_u is continuous. This means that ν is continuous.

3. Equicontinuous Transformation Groups

A transformation group (X, T, π) is said to be *equicontinuous* if π^T is equicontinuous. In this section, we obtain a necessary and sufficient condition for a transformation group (X, T, π) to be equicontinuous, related to the u.c. enveloping semigroup $E_u(X)$.

THEOREM 3.1. *(X, T, π) is equicontinuous iff $E_u(X)$ is precompact in the uniform space $(X^X, \mathscr{U}_u)$.*

Proof. It is clear by Theorem 1.1 and [4; Theorem 5.9].

REMARK 3.2. In fact, since X is compact Hausdorff, so is $E_u(X)$ if (X, T, π) is equicontinuous. In this case, there exist a minimal subset and an idempotent in the transformation group (E_u, T, ν) defined in Theorem 2.5. Moreover, M is a minimal subset of (E_u, T, ν) iff it is a minimal right ideal of E_u, and there exists an idempotent u in a minimal right ideal M of E_u. Furthermore, if $p\in M$ then $up=p$, and $Mu=\{pu\mid p\in M\}$ is a group with identity u.

LEMMA 3.3. *If Y is a compact Hausdorff space with a group structure such that all the left or right translations on Y are continuous, then it is a topological group.*

Proof. It is proved in [2; Theorem 2].

THEOREM 3.4. *If (X, T, π) is equicontinuous then $E_u(X)$ is a topological group.*

Proof. Since (X, T, π) is equicontinuous E_u is compact Hausdorff for the u.c. topology on X^X. There exists a minimal right ideal M of E_u and an idempotent u in M. Let x be any point of X and $y=u(x)$. Then $u(y)=u(x)$.

Since the map $\theta_{(x,y)}: p\mapsto(p(x), p(y))$ *of* E_u into $X\times X$ is continuous,

$$(u(x), u(y))\in\theta_{(x,y)}(E_u)=\theta_{(x,y)}(Cl(\pi^T))\subset Cl((x,y)T).$$

Thus $Cl((x,y)T)\cap\Delta(X)$ is nonempty. We have $x=y$ by the distality of (X, T, π). This implies that $u(x)=x$ for all x in X. Therefore,

$$p=up\in ME_u\subset M$$

for all p in E_u. Hence E_u is a minimal right ideal.

Furthermore, $E_u=E_u u$ clearly. So E_u is a group. Thus the proof is completed from Lemma 3.3, Remark 3.2 and Theorem 2.3.

COROLLARY 3.5. *If (X, T, π) is equicontinuous then $E_u(X)$ is a compact Hausdorff topological group consisting of homeomorphisms on X.*

Proof. It follows from Remark 3.2, Theorems 2.3 and 3.4.

THEOREM 3.6. *If T is a compact topological group then (X, T, π) is equicontinuous.*

Proof. Let U be any member of $\mathscr{U}$ and V a symmetric member of $\mathscr{U}$ with $V^2\subset U$. Since the map π is continuous, for every (x,t) in $X\times T$ there exist an open symmetric member $W_{(x,t)}$ of $\mathscr{U}$ and an open neighborhood $N_{(x,t)}$ of t in T such that $W_{(x,t)}{}^2[x]N_{(x,t)}$ is contained in $V[xt]$. Since the family $\{W_{(x,t)}[x]\times N_{(x,t)}\,|\,x\in X,\ t\in T\}$ is an open cover of the compact space $X\times T$. there exist (x_1,t_1), (x_2,t_2), $\cdots$, (x_n,t_n) in $X\times T$ such that $X\times T=\bigcup_{i=1}^{n}W_{(x_i,t_i)}[x_i]\times N_{(x_i,t_i)}$.

Now we set $W=\bigcap_{i=1}^{n}W_{(x_i,t_i)}$. Then W belongs to $\mathscr{U}$. Moreover,

$$(x,y)t=(xt,yt)\in V^2\subset U$$

for all (x, y) in W and t in T. This completes the proof.

COROLLARY 3.7. *If* (X, T, π) *is equicontinuous, then the map* μ : $(x, p) \mapsto p(x)$ *of* $X \times E_u$ *into* X *defines an equicontinuous transformation group* (X, E_u, μ).

Proof. It follows from Theorems 1.2 and 3.6.

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