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ON SOME PROPERTIES OF G-SPACES

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In a series of papers [1] and [2], D.H. Gottlieb introduced and studied the evaluation subgroups $G_m(X)$ of $\Pi_m(X)$. $G_m(X)$ is defined to be the set of all $\alpha\in\Pi_m(X)$ for which there is a representative f of α and a map $F: X\times S^m\to X$ of type $(1, f)$; that is, $F|X=1_X$, the identity mapping of X and $F|S^m=f$. A space X satisfying $G_m(X)=\Pi_m(X)$ for all m is called a G-space. Any H-space is a G-space, but the converse is not true [6].

In this paper, we define a weakly cyclic map and study some properties of weakly cyclic maps, and characterize G-spaces by these maps. We find a condition, under which H-spaces and G-spaces are equivalent. Also, we study homotopy groups of free mapping space X^{S^p} with a weakly cyclic maps as base point, and generalize a Koh's result.

Unless otherwise stated, we shall work in the category of spaces with base points and having the homotopy type of locally finite CW-complexes. All maps shall mean continuous functions. As usual, all maps and homotopies are to respect base points. The base points as well as the constant maps will be denoted by $*$. 1 (sometimes with decoration) will denote the identity function of space when it is clear from the context.

The folding map $\nabla: X\vee X\to X$ is given by $\nabla(x,*)=\nabla(*,x)=x$ for each $x\in X$.

Varadarajan [7] generalized $G_m(X)$ to $G(A, X)$ for any space A; $G(A, X)$ is the subset of $[A, X]$ containing all elements α which can be represented by a map $f: A\to X$ such that $\nabla(1\vee f): X\vee A\to X$ extends to a map $F: X\times A\to X$. The map $f: A\to X$, which is a representation of an element of $G(A, X)$ is called "cyclic". Lim called a map $f: A\to X$ is cyclic [5] if there exists a map $F: X\times A\to X$ such that the following diagram is homotopy commutative; that is $Fj\sim\nabla(1\vee f)$,

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$$\begin{array}{ccc} X\times A & \xrightarrow{\quad F \quad} & X \\ \uparrow j & (1\vee f) & \uparrow \nabla \\ X\vee A & \xrightarrow{\qquad\quad} & X\vee X \end{array}$$

where $j: X\vee A\to X\times A$ is the inclusion and $\nabla: X\vee X\to X$ is the folding map. However, in the category of spaces with base points and having the homotopy type of locally finite CW-complexes, the definition of Lim is equivalent to that of Varadarajan.

It is known [7] that if $f: A\to X$ is a cyclic map and $\theta: B\to A$ is an arbitrary map, then the composition map $f\circ\theta: B\to X$ is a cyclic map. In fact, the converse is also true. Therefore, a map $f: A\to X$ is a cyclic if and only if $f\circ\theta: B\to X$ is cyclic map for any space B and any map $\theta: B\to A$.

In [5], Lim showed that the following statements are equivalent;

1) X is an H-space
2) 1_X is cyclic
3) $G(A, X)=[A, X]$ for any space A.

Now we introduce weakly cyclic maps in order to study G-spaces.

DEFINITION 1. A map $f: A\to X$ is called *weakly cyclic* if for any sphere S^n and any map $\theta: S^n\to A$, $f\circ\theta: S^n\to X$ is cyclic. The set of all homotopy classes of weakly cyclic maps from A to X is denoted by $WG(A, X)$. In fact, $f: A\to X$ is a weakly cyclic if $f_*(\Pi_n(A))\subset G_n(X)$ for each n.

Clearly any cyclic map is weakly cyclic, but the converse does not hold (see Remark 5).

LEMMA 2. *Let $f: A\to X$ be a weakly cyclic map and $\theta: B\to A$ is an arbitrary map. Then $f\circ\theta: B\to X$ is weakly cyclic.*

LEMMA 3. *If $g: X\to Y$ has a right homotopy inverse, then g induces a map $g_*: WG(A, X)\to WG(A, Y)$ for any space A.*

Proof. For a weakly cyclic map $f: A\to X$, $(gf)_*(\Pi_n(A))= g_*f_*(\Pi_n(A))\subset g_*(G_n(X))\subset G_n(Y)$ for all n. The last part follows from Proposition 1.4 [2].

The next theorem shows that a G-space X may be characterized by several ways.

THEOREM 4. *The followings are pairwise equivalent;*
i) *X is a G-space*
ii) *1_X is weakly cyclic*
iii) *$WG(A, X) = [A, X]$ for any space A*
iv) *X is dominated by a G-space.*

Proof. i) implies ii). Since $\Pi_n(X) = G_n(X)$ for all n, 1_X is weakly cyclic.

ii) implies iii). Let A be any spaces and $[f] \in [A, X]$. Then $f = 1_X f$ is weakly cyclic by Lemma 2.

iii) implies i). Take $A = X$. 1_X is weakly cyclic, so that $\Pi_n(X) = G_n(X)$ for all n.

i) implies iv). It is clear. iv) implies ii). Let X be dominated by a G-space Y. Then there exists maps $k: X \to Y$ and $h: Y \to X$ such that $hk: X \to X$ is homotopic to 1_X. Since Y is a G-space, $k: X \to Y$ is weakly cyclic. Thus we have, by Lemma 3, $1_X \sim hk$ is weakly cyclic.

REMARK 5. Siegel [6] produced an example of 3-dimensional manifold T that is a G-space but not an H-space. Also, It is known [5] that X is an H-space if and only if 1_X is cyclic. Thus we have, by Theorem 4, the map 1_T that is weakly cyclic but not cyclic; that is, $G(A, X) \neq WG(A, X)$ in general. However, the following theorem shows that $G(A, X) = WG(A, X)$ under certain condition.

THEOREM 6. *Let X be any space. If A is domonated by a sphere S^m, then a map $f: A \to X$ is cyclic if and only if $f: A \to X$ is weakly cyclic.*

Proof. Clearly any cyclic map is weakly cyclic. Now, suppose that $f: A \to X$ is weakly cyclic. Since A is dominated by S^m, there exist maps $k: A \to S^m$ and $h: S^m \to A$ such that $hk: A \to A$ is homotopic to the map 1_A. Since $f: A \to X$ is weakly cyclic, $fh: S^m \to X$ is cyclic. Thus we have, by Lemma 1.3 [7], $f \sim (fh)k: A \to X$ is cyclic.

COROLLARY 7. *If a G-space X is dominated by a sphere S^m, then X is an H-space. In particular, the sphere S^m is G-space iff $m = 1, 3, 7$.*

Let X^{S^p} denote the space of free mappings from S^p to X with compact open topology. Then the evaluation map $w: X^{S^p} \to X$ given by $w(f) = f(*)$ is a fibre map. Now we study homotopy groups of free mapping space X^{S^p} with a weakly cyclic map as base point.

THEOREM 8. *If $f: S^p \to X$ is weakly cyclic, then $\Pi_q(X^{S^p}, f)/\Pi_{p+q}(X,*)$ is isomorphic to $\Pi_q(X,*)$ for all q, where $\Pi_{p+q}(X,*)$ is, of course, imbedded in $\Pi_q(X^{S^p}, f)$ isomorphically.*

Proof. For the fibration $w: X^{S^p} \to X$, let F be the fibre of w. Then we have a long exact sequence;

$$\cdots \to \Pi_q(F,f) \xrightarrow{i_*} \Pi_q(X^{S^p}, f) \xrightarrow{w_*} \Pi_q(X,*) \xrightarrow{\partial} \Pi_{q-1}(F,f) \to \cdots$$

Now, we show that w_* is an epimorphism. Let $g: S^q \to X$ be a representative of an element of $\Pi_q(X,*)$. Since $f: S^p \to X$ is a cyclic map, there exists a map $F: S^p \times X \to X$ such that $Fj = \nabla(f \vee 1_X)$. We take the map $G = F(1 \times g): S^p \times S^q \to X$. From the following commutative diagram;

$$\begin{array}{ccc} S^p \times S^q & \xrightarrow{\quad 1\times g \quad} & S^p \times X \\ \uparrow j & & \uparrow j \\ S^p \vee S^q & \xrightarrow{\quad 1\vee g \quad} & S^p \vee X, \end{array}$$

we have $Gj = F(1 \times g)j = Fj(1 \vee g) = \nabla(f \vee 1_X)(1 \vee g) = \nabla(f \vee g)$. Thus the map $G: S^p \times S^q \to X$ determines a map $\bar{G}: S^q \to X^{S^p}$, and observes that $\bar{G}(*) = f$. Therefore $\bar{G}$ represents an element of $\Pi_q(X^{S^p}, f)$. Since $w(\bar{G})(s) = \bar{G}(s)(*) = G(*, s) = g(s)$, we have $w_*([\bar{G}]) = [g]$. Thus the boundary maps are zero maps. Therefore we obtain a short exact sequence

$0 \to \Pi_q(F,f) \xrightarrow{i_*} \Pi_q(X^{S^p}, f) \xrightarrow{w_*} \Pi_q(X,*) \to 0$. The Hurewicz isomorphism $I_{[f]}: \Pi_q(F,f) \to \Pi_{p+1}(X,*)$ [3] completes the proof.

COROLLARY 9. *If X is a G-space, then for each $[f] \in \Pi_p(X,*)$, $\Pi_q(X^{S^p}, f)/\Pi_{p+q}(X,*)$ is isomorphic to $\Pi_q(X,*)$.*

Note that the above result was obtained by Koh [4] on the case of an H-space X.

THEOREM 10. *If $f: S^p \to X$ is weakly cyclic, then $\Pi_q(X^{S^p}, f)$ is isomorphic to $\Pi_{p+q}(X,*) \oplus \Pi_q(X,*)$ for all $q > 1$.*

Proof. In proof of Theorem 8, we have a short exact sequence

$$0 \to \Pi_q(F,f) \xrightarrow{i_*} \Pi_q(X^{S^p}, f) \xrightarrow{w_*} \Pi_q(X,*) \to 0.$$

We only show this sequence is split. Since $f: S^p \to X$ is cyclic, there exists a map $F: X \times S^p \to X$ such that $Fj = \nabla(1 \vee f)$. Now we define a map $k: (X, *) \to (X^{S^p}, f)$ by $k(x)(s) = F(x, s)$. Then we obtain a homomorphism $k_*: \Pi_q(X, *) \to \Pi_q(X^{S^p}, f)$ such that $w_* k_*$ is the identity on $\Pi_q(X, *)$. Therefore we have $\Pi_q(X^{S^p}, f) \cong \Pi_{p+q}(X, *) \oplus \Pi_q(X, *)$ for all $q > 1$.

Note that in the course of proof, we have shown that if $\Pi_1(X^{S^p}, f)$ and $\Pi_1(X, *)$ are abelian, then $\Pi_q(X^{S^p}, f)$ is isomorphic to $\Pi_{p+q}(X, *) \oplus \Pi_q(X, *)$ for all p, q, where $f: S^p \to X$ is a pointed weakly cyclic map. Since the constant map $*: S^p \to X$ is a weakly cyclic map, we can compute homotopy groups $\Pi_q(X^{S^p}, *)$ for all $q > 1$. Also, we know that if X is a G-space and $\Pi_{p+1}(X) = 0$, then for each $[f] \in \Pi_q(X, *)$, $\Pi_1(X^{S^p}, f)$ is an abelian group.

References

1. D.H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math., **87** (1965), 840-856.
2. D.H. Gottlieb, *Evaluation subgroups of hcmotopy groups*, Amer. J. Math., **91** (1969), 729-756.
3. I.M. James, *Reduced product spaces*, Ann. Math., **62** (1955), 170-179.
4. S.S. Koh, *Note on the properties of mapping spaces X^{S^p}*, Proc. Amer. Math. Soc., **11** (1960), 896-904.
5. K.L. Lim, *On cyclic maps*, J. Austral. Math. Soc., (Series A) **32** (1982), 349-357.
6. J. Siegel, *G-spaces, W-spaces, H-spaces*, Pac. J. Math., **31** (1969), 209-214.
7. K. Varadarajan, *Generalized Gottlieb grcups*, J. Indian. Math. Soc., **33** (1969), 141-164.

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