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SEMI-INVARIANT SUBMANIFOLDS OF A QUATERNIONIC PROJECTIVE SPACE

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0. Introduction

K. Yano and U-H. Ki([17]) introduced the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$—structure induced on a submanifold of a codimension 3 in an almost Hermitian manifold, and studied the submanifold with such a structure to define an almost contact metric structure on the submanifold.

By the way, Y. Tashiro and I.-B. Kim ([15]) have generalized the notion of the $(f, g, u, v, w, \lambda, \mu, \nu)$—structure by defining the so-called metric compound structure in a submanifold of an almost Hermitian manifold.

In the present paper, we study semi-invariant submanifolds of a quaternionic projective space admitting an almost contact metric 3-structure, which will be called an almost contact metric compound 3-structure.

Manifolds, submanifolds, geometric objects and mappings discussed in this paper are assumed to be of c^∞. We use throughout this paper the systems of indices as follows:

$$\begin{aligned} &\kappa, \mu, \nu, \lambda, \cdots = 1, 2 \cdots, 4m+3; \quad \alpha, \beta, \gamma, \delta, \cdots = 1, 2, \cdots, n+3; \\ &A, B, C, D, \cdots = 1, 2, \cdots, 4m; \quad h, i, j, k, \cdots = 1, 2, \cdots, n; \\ &w, x, y, z, \cdots 1^*, 2^*, \cdots, p^*, \quad n+p=4m; \quad r, s, t, \cdots = 1, 2, 3. \end{aligned}$$

The summation covention will be used with respect to those systems of indices.

1. Preliminaries

First, we recall the definition of a quaternionic Kaehlerian structure given by S. Ishihara ([5]). Let $\widetilde{M}$ be a $4m$-dimensional differentiable manifold and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over $\tilde{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\{\tilde{U};x^A\}$, there is a local base $\{F, G, H\}$ of V such that

$$(1.1)\qquad \begin{aligned} &F_B{}^A F_C{}^B=-\delta_C{}^A,\ G_B{}^A G_C{}^B=-\delta_C{}^A,\ H_B{}^A H_C{}^B=-\delta_C{}^A,\\ &F_B{}^A G_C{}^B=-G_B{}^A F_C{}^B=H_C{}^A,\ G_B{}^A H_C{}^B=-H_B{}^A G_C{}^B=F_C{}^A,\\ &H_B{}^A F_C{}^B=-F_B{}^A H_C{}^B=G_C{}^A,\end{aligned}$$

$F_B{}^A$, $G_B{}^A$ and $H_B{}^A$ denoting the components of F, G and H in $\tilde{U}$ respectively.

(b) There is a Riemannian metric tensor g_{AB} such that

$$F_{AB}=-F_{BA},\ G_{AB}=-G_{BA},\ H_{AB}=-H_{BA},$$

where $F_{AB}=g_{CB}F_A{}^C$, $G_{AB}=g_{CB}G_A{}^C$ and $H_{AB}=g_{CB}H_A{}^C$.

(c) For the Riemannian connection $\tilde{\nabla}$ of $(\tilde{M}, g)$,

$$(1.2)\qquad \begin{aligned} &\tilde{\nabla}_C F_B{}^A=w_C G_B{}^A-v_C H_B{}^A,\ \tilde{\nabla}_C G_B{}^A=-w_C F_B{}^A+u_C H_B{}^A,\\ &\tilde{\nabla}_C H_B{}^A=v_C F_B{}^A-u_C G_B{}^A,\end{aligned}$$

where $u=u_A dx^A$, $v=v_A dx^A$ and $w=w_A dx^A$ are certain local 1-forms defined in $\tilde{U}$. Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in $\tilde{U}$ and $(\tilde{M}, g, V)$ or $\tilde{M}$ is called a *quaternionic Kaehlerian manifold* and (g, V) a *quaternionic Kaehlerian structure*.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U;x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i: M\to\tilde{M}$. We identify $i(M)$ with M itself and represent the immersion locally by

$$(1.3)\qquad x^A=x^A(x^h).$$

We now put $B_i{}^A=\partial_i x^A$ $(\partial_i=\partial/\partial x^i)$. Then $B_i{}^A$ are n linearly independent vectors of $\tilde{M}$ tangent to M. And denote by $C_x{}^A$ mutually orthogonal unit normal vector fields of M. Then we have $g_{CB}B_i{}^B C_x{}^C=0$ and the metric tensor of the normal bundle of M is given by $g_{zy}=g_{CB}C_z{}^C C_y{}^B=\delta_{zy}$. Therefore, vector fields $B_i{}^A$ and $C_x{}^A$ span the tangent space $T_p(\tilde{M})$ of $\tilde{M}$ at every point p of M. The metric tensor g of M induced from that of $\tilde{M}$ is given by

$$(1.4)\qquad g_{ij}=g_{CB}B_j{}^C B_i{}^B$$

since the immersion is isometric.

The transform of the tangent vectors $B_j{}^A$ and the normal vectors $C_x{}^A$ to M by F, G and H are expressed in the form:

$$F_B{}^A B_j{}^B = \phi_j{}^h B_h{}^A + \phi_j{}^x C_x{}^A, \quad F_B{}^A C_x{}^B = -\phi_x{}^h B_h{}^A + \phi_x{}^y C_y{}^A, \tag{1.5}$$

$$G_B{}^A B_j{}^B = \psi_j{}^h B_h{}^A + \psi_j{}^x C_x{}^A, \quad G_B{}^A C_x{}^B = -\psi_x{}^h B_h{}^A + \psi_x{}^y C_y{}^A, \tag{1.6}$$

$$H_B{}^A B_j{}^B = \theta_j{}^h B_h{}^A + \theta_j{}^x C_x{}^A, \quad H_B{}^A C_x{}^B = -\theta_x{}^h B_h{}^A + \theta_x{}^y C_y{}^A, \tag{1.7}$$

respectively, where $\phi_j{}^h, \psi_j{}^h$ and $\theta_j{}^h$ are components of a tensor fields of type (1, 1), $\phi_j{}^x, \psi_j{}^x$ and $\theta_j{}^x$ 1-forms for each fixed $x, \phi_x{}^h, \psi_x{}^h$ and $\theta_x{}^h$ vector fields associated with $\phi_j{}^x$, $\psi_j{}^x$ and $\theta_j{}^x$ given by $\phi_x{}^h = \phi_j{}^y g^{jh} g_{yx}$, $\psi_x{}^h = \psi_j{}^y g^{jh} g_{yx}$, $\theta_x{}^h = \theta_j{}^y g^{jh} g_{yx}$, and $\phi_x{}^y$, $\psi_x{}^y$ and $\theta_x{}^y$ are functions for fixed indices x and y. We can easily find

$$\begin{aligned} &\phi_{ji} = -\phi_{ij}, \quad \phi_{jx} = \phi_{xj}, \quad \phi_{xy} = -\phi_{yx}, \\ &\psi_{ji} = -\psi_{ij}, \quad \psi_{jx} = \psi_{xj}, \quad \psi_{xy} = -\psi_{yx}, \\ &\theta_{ji} = -\theta_{ij}, \quad \theta_{jx} = \theta_{xj}, \quad \theta_{xy} = -\theta_{yx}, \end{aligned} \tag{1.8}$$

where $\phi_{ji} = \phi_j{}^h g_{hi}$, $\phi_{xj} = \phi_x{}^h g_{hj}$, $\phi_{jx} = \phi_j{}^y g_{yx}$, $\phi_{xy} = \phi_x{}^z g_{zy}$, etc.

Applying F to (1.5), G to (1.6), and H to (1.7) respectively, and using (1.1) and these expressions, we have

$$\begin{aligned} &\phi_j{}^i \phi_i{}^h = -\delta_j{}^h + \phi_j{}^x \phi_x{}^h, \quad \phi_j{}^h \phi_h{}^y + \phi_j{}^x \phi_x{}^y = 0, \\ &\phi_x{}^h \phi_h{}^i + \phi_x{}^y \phi_y{}^i = 0, \quad \phi_y{}^z \phi_z{}^x = -\delta_y{}^x + \phi_y{}^h \phi_h{}^x, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &\psi_j{}^i \psi_i{}^h = -\delta_j{}^h + \psi_j{}^x \psi_x{}^h, \quad \psi_j{}^h \psi_h{}^y + \psi_j{}^x \psi_x{}^y = 0, \\ &\psi_x{}^h \psi_h{}^i + \psi_x{}^y \psi_y{}^i = 0, \quad \psi_y{}^z \psi_z{}^x = -\delta_y{}^x + \psi_y{}^h \psi_h{}^x, \end{aligned} \tag{1.10}$$

$$\begin{aligned} &\theta_j{}^i \theta_i{}^h = -\delta_j{}^h + \theta_j{}^x \theta_x{}^h, \quad \theta_j{}^h \theta_h{}^y + \theta_j{}^x \theta_x{}^y = 0, \\ &\theta_x{}^h \theta_h{}^i + \theta_x{}^y \theta_y{}^i = 0, \quad \theta_y{}^z \theta_z{}^x = -\delta_y{}^x + \theta_y{}^h \theta_h{}^x. \end{aligned} \tag{1.11}$$

Applying G, H to (1.5), and H to (1.6) respectively, and using (1.1), (1.5), we have

$$\begin{aligned} &\phi_j{}^h \psi_h{}^i = -\theta_j{}^i + \phi_j{}^x \psi_x{}^i, \quad \phi_x{}^h \psi_h{}^i = -\theta_x{}^i - \phi_x{}^y \psi_y{}^i, \\ &\phi_j{}^h \psi_h{}^y = -\theta_j{}^y - \phi_j{}^x \psi_x{}^y, \quad \phi_x{}^h \psi_h{}^y = \theta_x{}^y + \phi_x{}^z \psi_z{}^y, \\ &\phi_j{}^h \theta_h{}^i = \psi_j{}^i + \phi_j{}^x \theta_x{}^i, \quad \phi_x{}^h \theta_h{}^i = \psi_x{}^i - \phi_x{}^y \theta_y{}^i, \\ &\phi_j{}^h \theta_h{}^y = \psi_j{}^y - \phi_j{}^x \theta_x{}^y, \quad \phi_x{}^h \theta_h{}^y = -\psi_x{}^y + \phi_x{}^z \theta_z{}^y, \\ &\psi_j{}^h \theta_h{}^i = -\phi_j{}^i + \psi_j{}^x \theta_x{}^i, \quad \psi_x{}^h \theta_h{}^i = -\phi_x{}^i - \psi_x{}^y \theta_y{}^i, \\ &\psi_j{}^h \theta_h{}^y = -\phi_j{}^y - \psi_j{}^x \theta_x{}^y, \quad \psi_x{}^h \theta_h{}^y = \phi_x{}^y + \psi_x{}^z \theta_z{}^y, \\ &\psi_j{}^h \phi_h{}^i = \theta_j{}^i + \psi_j{}^x \phi_x{}^i, \quad \psi_x{}^h \phi_h{}^i = \theta_x{}^i - \psi_x{}^y \phi_y{}^i, \\ &\psi_j{}^h \phi_h{}^y = \theta_j{}^y - \psi_j{}^x \phi_x{}^y, \quad \psi_x{}^h \phi_h{}^y = -\theta_x{}^y + \psi_x{}^z \phi_z{}^y, \end{aligned} \tag{1.12}$$

$$\theta_j{}^h\phi_h{}^i=-\psi_j{}^i+\theta_j{}^x\phi_x{}^i,\quad \theta_x{}^h\phi_h{}^i=-\psi_x{}^i-\theta_x{}^y\phi_y{}^i,$$
$$\theta_j{}^h\phi_h{}^y=-\psi_j{}^y-\theta_j{}^x\phi_x{}^y,\quad \theta_x{}^h\phi_h{}^y=\psi_x{}^y+\theta_x{}^z\phi_z{}^y,$$
$$\theta_j{}^h\psi_h{}^i=\phi_j{}^i+\theta_j{}^x\psi_x{}^i,\quad \theta_x{}^h\psi_h{}^i=\phi_x{}^i-\theta_x{}^y\psi_y{}^i,$$
$$\theta_j{}^h\psi_h{}^y=\phi_j{}^y-\theta_j{}^x\psi_x{}^y,\quad \theta_x{}^h\psi_h{}^y=-\phi_x{}^y+\theta_x{}^z\psi_z{}^y.$$

From $F_{AB}=g_{CB}F_A{}^C$, $G_{AB}=g_{CB}G_A{}^C$, $H_{AB}=g_{CB}H_A{}^C$, (1.4), (1.5), (1.6) and (1.7), we have

$$(1.13)\qquad \phi_j{}^h\phi_i{}^k g_{hk}=g_{ji}-\phi_j{}^x\phi_{ix},\quad \psi_j{}^h\psi_i{}^k g_{hk}=g_{ji}-\psi_j{}^x\psi_{ix},$$
$$\theta_j{}^h\theta_i{}^k g_{hk}=g_{ji}-\theta_j{}^x\theta_{ix}.$$

Now, removing the quaternionic Kaehlerian ambient manifold $\widetilde{M}$, we suppose that an n-dimensional Riemannian manifold M admits a metric tensor g_{ji}, tensor fields $\phi_j{}^h$, $\psi_j{}^h$ and $\theta_j{}^h$ of type (1, 1), p vector fields $\phi_x{}^h$, $\psi_x{}^h$ and $\theta_x{}^h$, p 1-forms $\phi_j{}^x$, $\psi_j{}^x$ and $\theta_j{}^x$ and $p(p-1)/2$ scalar fields ϕ_{xy}, ψ_{xy} and θ_{xy} satisfying the relationships (1.9)$\sim$(1.13). Such a set $(\phi_j{}^h,\ \phi_x{}^h,\ \phi_x{}^y,\ \psi_j{}^h,\ \psi_x{}^h,\ \psi_x{}^y,\ \theta_j{}^h,\ \theta_x{}^h,\ \theta_x{}^y,\ g_{ji})$ is said to be a *metric compound* 3-*structure* on M.

We suppose that M admits an almost contact metric 3-structure. Then we have

$$(1.14)\qquad \phi_j{}^i\phi_i{}^h=-\delta_j^h+p_jp^h,\quad \phi_j{}^ip_i=0,\quad \phi_j{}^hp^j=0,\quad p_ip^i=1,$$
$$\psi_j{}^i\psi_i{}^h=-\delta_j^h+q_jq^h,\quad \psi_j{}^iq_i=0,\quad \psi_j{}^hq^j=0,\quad q_jq^j=1,$$
$$\theta_j{}^i\theta_i{}^h=-\delta_j^h+r_jr^h,\quad \theta_j{}^ir_i=0,\quad \theta_j{}^hr^j=0,\quad r_jr^j=1,$$
$$\psi_j{}^i\theta_i{}^h=-\phi_j{}^h+q_jr^h,\quad \theta_j{}^i\psi_i{}^h=\phi_j{}^h+r_jq^h,$$
$$\phi_j{}^i\psi_i{}^h=-\theta_j{}^h+p_jq^h,\quad \psi_j{}^i\phi_i{}^h=\theta_j{}^h+q_jp^h,$$
$$\theta_j{}^i\phi_i{}^h=-\psi_j{}^h+r_jp^h,\quad \phi_j{}^i\theta_i{}^h=\psi_j{}^h+p_jr^h,$$
$$\phi_j{}^hq_h=-r_j,\quad \phi_j{}^hr_h=q_j,\quad \psi_j{}^hr_h=-p_j,$$
$$\psi_j{}^hp_h=r_j,\quad \theta_j{}^hp_h=-q_j,\quad \theta_j{}^hq_h=p_j,$$
$$\phi_{ji}=-\phi_{ij},\quad \psi_{ji}=-\psi_{ij},\quad \theta_{ji}=-\theta_{ij},$$

where p_j, q_j and r_j are 1-forms and p^h, q^h and r^h vector fields associated with p_j, q_j and r_j given by $p^h=g^{hj}p_j$, $q^h=g^{hj}q_j$ and $r^h=g^{hj}r_j$ on M.

In this case, we know that the dimension n of M is odd and the rank of $(\phi_j{}^i)$, $(\psi_j{}^i)$ and $(\theta_j{}^i)$ are equal to $n-1$.

Comparing with (1.13) and (1.14), we have

$$(1.15)\qquad \phi_j{}^x\phi_{ix}=p_jp_i,\quad \psi_j{}^x\psi_{ix}=q_jq_i,\quad \theta_j{}^x\theta_{ix}=r_jr_i.$$

These equations show that the product of the matrices $(\phi_j{}^x)$, $(\psi_j{}^x)$, and

$(\theta_j{}^x)$ with their respective transposes is of rank 1 and hence the rank of all the matrices $(\phi_j{}^x)$, $(\psi_j{}^x)$ and $(\theta_j{}^x)$ is 1. Therefore, we may put

$$(1.16)\qquad \phi_j{}^x=\nu_1{}^x p_j,\quad \psi_j{}^x=\nu_2{}^x q_j,\quad \theta_j{}^x=\nu_3{}^x r_j,$$

where $\nu_i{}^x$ $(i=1,2,3)$ are certain scalar fields for each i. Since $\phi_{jx}\phi^{jx}=p_jp^j=1$, $\psi_{jx}\psi^{jx}=q_jq^j=1$, and $\theta_{jx}\theta^{jx}=1$, we get

$$(1.17)\qquad \nu_1{}^x=\nu_2{}^x=\nu_3{}^x,\quad \nu_i{}^x\nu_{jx}=1\quad (i,j=1,2,3)$$

by virtue of (1.12), (1.14) and (1.15). Putting $\nu^x=\nu_1{}^x=\nu_2{}^x=\nu_3{}^x$ and using (1.9), (1.10) and (1.11), we have

$$(1.18)\qquad \phi_x{}^y\nu^x=0,\quad \psi_x{}^y\nu^x=0,\quad \theta_x{}^y\nu^x=0,$$

$$(1.19)\qquad \phi_y{}^z\phi_z{}^x=-\delta_y{}^x+\nu_y\nu^x,\quad \psi_y{}^z\psi_z{}^x=-\delta_y{}^x+\nu_y\nu^x,\quad \theta_y{}^z\theta_z{}^x=-\delta_y{}^x+\nu_y\nu^x.$$

From (1.12) and (1.14), we have

$$(1.20)\qquad \begin{aligned}&\phi_x{}^z\psi_z{}^y=-\theta_x{}^y,\quad \phi_x{}^z\theta_z{}^y=\psi_x{}^y,\quad \psi_x{}^z\theta_z{}^y=-\phi_x{}^y,\\ &\psi_x{}^z\phi_z{}^y=\theta_x{}^y,\quad \theta_x{}^z\phi_z{}^y=-\psi_x{}^y,\quad \theta_x{}^z\psi_z{}^y=\phi_x{}^y.\end{aligned}$$

A set $(\phi_y{}^x,\ \psi_y{}^x,\ \theta_y{}^x,\ g_{yx},\ \nu^x)$ satisfying the relationships (1.17)$\sim$(1.20) is said to be a *semi-almost contact metric 3-structure* on R^p, and consequently we see that the dimension p of R^p is odd.

Conversely, assuming that a semi-almost contact metric 3-structure $(\phi_y{}^x,\ \psi_y{}^x,\ \theta_y{}^x,\ g_{xy},\ \nu^x)$ on R^p is admitted, we can prove that the metric compound 3-structure $(\phi_j{}^h,\ \phi_x{}^h,\ \phi_x{}^y,\ \psi_j{}^h,\ \psi_x{}^h,\ \psi_x{}^y,\ \theta_j{}^h,\ \theta_x{}^h,\ \theta_x{}^y,\ g_{ji})$ induces an almost contact metric 3-structure $(\phi_j{}^h,\ \psi_j{}^h,\ \theta_j{}^h,\ g_{ji},\ p^h,\ q^h,\ r^h)$ on M.

Thus we have

THEOREM 1.1. Let $(\phi_j{}^h, \phi_x{}^h, \phi_x{}^y, \psi_j{}^h, \psi_x{}^h, \psi_x{}^y, \theta_j{}^h, \theta_x{}^h, \theta_x{}^y, g_{ji})$ *be a metric compound 3-structure on M. In order that* $\phi_j{}^h, \psi_j{}^h, \theta_j{}^h$ *and* g_{ji} *constitute an almost contact metric 3-structure* $(\phi_j{}^h, \psi_j{}^h, \theta_j{}^h, g_{ji}, p^h, q^h, r^h)$ *on M, it is necessary and sufficient that* $\phi_y{}^x, \psi_y{}^x, \theta_y{}^x$ *and* g_{yx} *constitute a semi-almost contact matric 3-structure* $(\phi_y{}^x, \psi_y{}^x, \theta_y{}^x, g_{yx}, \nu^x)$ *on* R^p.

From above discussions we also have

THEOREM 1.2. *In order for a metric compound 3-structure* $(\phi_j{}^h, \phi_x{}^h, \phi_x{}^y, \psi_j{}^h, \psi_x{}^h, \psi_x{}^y, \theta_j{}^h, \theta_x{}^h, \theta_x{}^y, g_{ji})$ *to be almost contact metric 3-structure, it is* *necessary and sufficient that the matrices* $(\phi_x{}^h)$, $(\psi_x{}^h)$ *and* $(\theta_x{}^h)$ *are of rank* 1, *that is, the p vector fields* $\phi_x{}^h$, $\psi_x{}^h$, *and* $\theta_x{}^h$ *are all parallel to* p^h, q^h *and* r^h *respectively.*

A metric compound 3-structure admitting an almost contact metric 3-structure is said to be an *almost contact metric compound 3-structure* on M.

2. Submanifolds of codimension p of a quaternionic Kaehlerian manifold

In this section we assume that n-dimensional submanifold M of codimension p of a quaternionic Kaehlerian manifold $\widetilde{M}$ admits an almost contact metric compound 3-structure, and consequently $(\phi_j{}^h, \psi_j{}^h, \theta_j{}^h, g_{ji}, p^h, q^h, r^h)$ defines an almost contact metric 3-structure. So, (1.14) is valid.

The vector field N^A defined by

$$N^A = \nu^x C_x{}^A \tag{2.1}$$

is unit normal to M because $g_{CB}C_x{}^C C_y{}^B = \delta_{xy}$ and $\nu_x\nu^x = 1$.

If we transform the tangent vectors $B_i{}^A$ and the unit normal vector N^A by F, G and H, then we have

$$F_B{}^A B_j{}^B = \phi_j{}^h B_h{}^A + p_j N^A, \quad F_B{}^A N^B = -p^h B_h{}^A, \tag{2.2}$$

$$G_B{}^A B_j{}^B = \psi_j{}^h B_h{}^A + q_j N^A, \quad G_B{}^A N^B = -q^h B_h{}^A, \tag{2.3}$$

$$H_B{}^A B_j{}^B = \theta_j{}^h B_h{}^A + r_j N^A, \quad H_B{}^A N^B = -r^h B_h{}^A \tag{2.4}$$

respectively because of (1.5)$\sim$(1.7), (1.16), (1.18) and (2.1).

It is well known that the submanifold M of a quaternionic Kaehlerian manifold satisfying (2.2)$\sim$(2.4) is semi-invariant with respect to N^A and we call N^A the distinguished normal to M ([1], [14]).

Now, we take N^A as $C_{1*}{}^A$. Then we have from (2.1) that $\nu^{1*} = 1$ and $\nu^{(x)} = 0$, where here and in the sequel, (x) runs over the range $\{2^*, \cdots, p^*\}$. For the convenience of notation, we write C^A instead of $C_{1*}{}^A$. Then we can represent (2.2)$\sim$(2.4) respectively as follows:

$$F_B{}^A B_j{}^B = \phi_j{}^h B_h{}^A + p_j C^A, \quad F_B{}^A C^B = -p^h B_h{}^A, \tag{2.5}$$

$$G_B{}^A B_j{}^B = \psi_j{}^h B_h{}^A + q_j C^A, \quad G_B{}^A C^B = -q^h B_h{}^A, \tag{2.6}$$

$$H_B{}^A B_j{}^B = \theta_j{}^h B_h{}^A + r_j C^A, \quad H_B{}^A C^B = -r^h B_h{}^A. \tag{2.7}$$

Taking account of (1.16), (1.18) and the fact that $\nu^{1*}=1$ and $\nu^{(x)}=0$, we find from (1.5), (1.6) and (1.7)

$$F_B{}^A C_{(x)}{}^B=\phi_{(x)}{}^{(y)}C_{(y)}{}^A,\quad G_B{}^A C_{(x)}{}^B=\psi_{(x)}{}^{(y)}C_{(y)}{}^A,\quad H_B{}^A C_{(x)}{}^B=\theta_{(x)}{}^{(y)}C_{(y)}{}^A \tag{2.8}$$

respectively. Then, by applying F, G and H to (2.8), it follows

$$\phi_{(x)}{}^{(z)}\phi_{(z)}{}^{(y)}=-\delta_{(x)}{}^{(y)},\quad \phi_{(x)}{}^{(z)}\psi_{(z)}{}^{(y)}=-\theta_{(x)}{}^{(y)},\quad \phi_{(x)}{}^{(z)}\theta_{(z)}{}^{(y)}=\psi_{(x)}{}^{(y)}, \tag{2.9}$$

$$\psi_{(x)}{}^{(z)}\psi_{(z)}{}^{(y)}=-\delta_{(x)}{}^{(y)},\quad \psi_{(x)}{}^{(z)}\theta_{(z)}{}^{(y)}=-\phi_{(x)}{}^{(y)},\quad \psi_{(x)}{}^{(z)}\phi_{(z)}{}^{(y)}=\theta_{(x)}{}^{(y)}, \tag{2.10}$$

$$\theta_{(x)}{}^{(z)}\theta_{(z)}{}^{(y)}=-\delta_{(x)}{}^{(y)},\quad \theta_{(x)}{}^{(z)}\phi_{(z)}{}^{(y)}=-\psi_{(x)}{}^{(y)},\quad \theta_{(x)}{}^{(z)}\psi_{(z)}{}^{(y)}=\phi_{(x)}{}^{(y)} \tag{2.11}$$

respectively.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to the fundamental tensor g_{ji}, we have the equations of Gauss for M

$$\nabla_j B_i{}^A=A_{ji}C^A+A_{ji}{}^{(x)}C_{(x)}{}^A, \tag{2.12}$$

where A_{ji} and $A_{ji}{}^{(x)}$ are the second fundamental tensors with respect to normal vector fields C^A and $C_{(x)}{}^A$ respectively and those of Weingarten

$$\nabla_j C^A=-A_j{}^h B_h{}^A+l_j{}^{(x)}C_{(x)}{}^A, \tag{2.13}$$

$$\nabla_j C_{(x)}{}^A=-A_j{}^h{}_{(x)}B_h{}^A-l_{j(x)}C^A+l_{j(x)}{}^{(y)}C_{(y)}{}^A, \tag{2.14}$$

where $A_j{}^h=g^{hi}A_{ji}$ $A_j{}^h{}_{(x)}=g^{hi}g_{(y)(x)}A_{ji}{}^{(y)}=A_{ji(x)}g^{hi}$, $l_j{}^{(x)}$ and $l_{j(x)}{}^{(y)}$ are the third fundamental tensors, $l_{j(x)}=l_j{}^{(y)}g_{(y)(x)}$.

Putting $l_{j(x)(y)}=l_{j(x)}{}^{(z)}g_{(z)(y)}$, we can easily verify $l_{j(x)(y)}=-l_{j(y)(x)}$ since $C_{(x)}{}^A$ are mutually orthogonal.

We now assume that the ambient manifold $\widetilde{M}$ is a quaternionic Kaehlerian manifold, that is, (1.2) holds.

Differentiating (2.5) ~ (2.8) covariantly and using (2.12) ~ (2.14) and these equations and putting $u_j=u_A B_j{}^A$, $v_j=v_A B_j{}^A$ and $w_j=w_A B_j{}^A$, we can easily find

$$\begin{aligned}\nabla_j\phi_i{}^h&=-A_{ji}p^h+A_j{}^h p_i+\psi_i{}^h w_j-\theta_i{}^h v_j,\\ \nabla_j\psi_i{}^h&=-A_{ji}q^h+A_j{}^h q_i+\theta_i{}^h u_j-\phi_i{}^h w_j,\\ \nabla_j\theta_i{}^h&=-A_{ji}r_h+A_j{}^h r_i+\phi_i{}^h v_j-\psi_i{}^h u_j,\end{aligned} \tag{2.15}$$

$$(2.16)\quad \begin{aligned} &\nabla_j p_i = -A_{jh}\phi_i{}^h + w_j q_i - v_j r_i, \quad \nabla_j p^h = A_j{}^i\phi_i{}^h + w_j q^h - v_j r^h,\\ &\nabla_j q_i = -A_{jh}\psi_i{}^h + u_j r_i - w_j p_i, \quad \nabla_j q^h = A_j{}^i\psi_i{}^h + u_j r^h - w_j p^h,\\ &\nabla_j r_i = -A_{jh}\theta_i{}^h + v_j p_i - u_j q_i, \quad \nabla_j r^h = A_j{}^i\theta_i{}^h + v_j p^h - u_j q^h, \end{aligned}$$

$$(2.17)\quad \begin{aligned} A_{ji}{}^{(x)}\phi_{(x)}{}^{(y)} &= \phi_i{}^h A_{jh}{}^{(y)} + l_j{}^{(y)} p_i,\\ A_{ji}{}^{(x)}\psi_{(x)}{}^{(y)} &= \psi_i{}^h A_{jh}{}^{(y)} + l_j{}^{(y)} q_i,\\ A_{ji}{}^{(x)}\theta_{(x)}{}^{(y)} &= \theta_i{}^h A_{jh}{}^{(y)} + l_j{}^{(y)} r_i, \end{aligned}$$

$$(2.18)\quad \begin{aligned} &A_{jh}{}^{(x)}p^h = -l_j{}^{(y)}\phi_{(y)}{}^{(x)}, \quad A_{jh}{}^{(x)}q^h = -l_j{}^{(y)}\psi_{(y)}{}^{(x)},\\ &A_{jh}{}^{(x)}r^h = -l_j{}^{(y)}\theta_{(y)}{}^{(x)}, \end{aligned}$$

$$(2.19)\quad \begin{aligned} \nabla_j\phi_{(x)}{}^{(y)} &= l_{j(x)}{}^{(z)}\phi_{(z)}{}^{(y)} - \phi_{(x)}{}^{(z)} l_{j(z)}{}^{(y)} + w_j\psi_{(x)}{}^{(y)} - v_j\theta_{(x)}{}^{(y)},\\ \nabla_j\psi_{(x)}{}^{(y)} &= l_{j(x)}{}^{(z)}\psi_{(z)}{}^{(y)} - \psi_{(x)}{}^{(z)} l_{j(z)}{}^{(y)} + u_j\theta_{(x)}{}^{(y)} - w_j\phi_{(x)}{}^{(y)},\\ \nabla_j\theta_{(x)}{}^{(y)} &= l_{j(x)}{}^{(z)}\theta_{(z)}{}^{(y)} - \theta_{(x)}{}^{(z)} l_{j(z)}{}^{(y)} + v_j\phi_{(x)}{}^{(y)} - u_j\psi_{(x)}{}^{(y)}. \end{aligned}$$

If we transvect (2.17) with $\phi_{(y)}{}^{(z)}$, $\psi_{(y)}{}^{(z)}$ and $\theta_{(y)}{}^{(z)}$ respectively and using (2.9), (2.10) and (2.11), then we obtain

$$(2.20)\quad \begin{aligned} A_{ji}{}^{(z)} &= -A_{jh}{}^{(y)}\phi_i{}^h\phi_{(y)}{}^{(z)} - l_j{}^{(y)}p_i\phi_{(y)}{}^{(z)},\\ A_{ji}{}^{(z)} &= -A_{jh}{}^{(y)}\psi_i{}^h\psi_{(y)}{}^{(z)} - l_j{}^{(y)}q_i\psi_{(y)}{}^{(z)},\\ A_{ji}{}^{(z)} &= -A_{jh}{}^{(y)}\theta_i{}^h\theta_{(y)}{}^{(z)} - l_j{}^{(y)}r_i\theta_{(y)}{}^{(z)}, \end{aligned}$$

from which,

$$(2.21)\quad A^{(z)} = -p^h l_h{}^{(y)}\phi_{(y)}{}^{(z)} = -q^h l_h{}^{(y)}\psi_{(y)}{}^{(z)} = -r^h l_h{}^{(y)}\theta_{(y)}{}^{(z)},$$

where $A^{(z)} = g^{ji}A_{ji}{}^{(z)}$.

Also, transvecting (2.18) with $\phi_{(x)}{}^{(z)}$, $\psi_{(x)}{}^{(z)}$ and $\theta_{(x)}{}^{(z)}$ respectively and using (2.9)～(2.11), we find

$$(2.22)\quad l_j{}^{(z)} = A_{jh}{}^{(x)}p^h\phi_{(x)}{}^{(z)} = A_{jh}{}^{(x)}q^h\psi_{(x)}{}^{(z)} = A_{jh}{}^{(x)}r^h\theta_{(x)}{}^{(z)}.$$

The equations of Gauss for M in a quaternionic Kaehlerian manifold $\widetilde{M}$ are given by

$$(2.23)\quad \begin{aligned} K_{kji}{}^h = {} & K_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B B_A{}^h + A_k{}^h A_{ji} - A_j{}^h A_{ki}\\ & + A_k{}^h{}_{(x)} A_{ji}{}^{(x)} - A_j{}^h{}_{(x)} A_{ki}{}^{(x)}, \end{aligned}$$

where $K_{DCB}{}^A$ and $K_{kji}{}^h$ are the Riemann-Christoffel curvature tensors of $\widetilde{M}$ and M respectively, and we have put $B_A{}^h = B_i{}^B g^{hi} g_{AB}$.

We now assume that the ambient manifold $\widetilde{M}$ is a $4m$-dimensional quaternionic Kaehlerian manifold of constant Q-sectional curvature c. It is well known that its curvature tensor has components of the form

$$
\begin{aligned}
(2.24)\qquad K_{DCB}{}^{A}=\frac{c}{4}(&\delta_{D}{}^{A}g_{CB}-\delta_{C}{}^{A}g_{DB}+F_{D}{}^{A}F_{CB}-F_{C}{}^{A}F_{DB}\\
&-2F_{DC}F_{B}{}^{A}+G_{D}{}^{A}G_{CB}-G_{C}{}^{A}G_{DB}-2G_{DC}G_{B}{}^{A}\\
&+H_{D}{}^{A}H_{CB}-H_{C}{}^{A}H_{DB}-2H_{DC}H_{B}{}^{A}),
\end{aligned}
$$

where c is necessary a constant, provided $m\geqq 2$ ([5]). Substituting (2.24) into (2.23) and using (1.4), (2.5), (2.6) and (2.7), we can see that

$$
\begin{aligned}
(2.25)\qquad K_{kji}{}^{h}=\frac{c}{4}(&\delta_{k}{}^{h}g_{ji}-\delta_{j}{}^{h}g_{ki}+\phi_{k}{}^{h}\phi_{ji}-\phi_{j}{}^{h}\phi_{ki}-2\phi_{kj}\phi_{i}{}^{h}\\
&+\psi_{k}{}^{h}\psi_{ji}-\psi_{j}{}^{h}\psi_{ki}-2\psi_{kj}\psi_{i}{}^{h}+\theta_{k}{}^{h}\theta_{ji}-\theta_{j}{}^{h}\theta_{ki}-2\theta_{kj}\theta_{i}{}^{h})\\
&+A_{k}{}^{h}A_{ji}-A_{j}{}^{h}A_{ki}+A_{k}{}^{h}{}_{(x)}A_{ji}{}^{(x)}-A_{j}{}^{h}{}_{(x)}A_{ki}{}^{(x)}.
\end{aligned}
$$

By taking account of (2.5)~(2.8), (2.13), (2.14) and (2.24), we have the equations of Codazzi:

$$
\begin{aligned}
(2.26)\qquad &\nabla_{k}A_{ji}-\nabla_{j}A_{ki}-l_{k(x)}A_{ji}{}^{(x)}+l_{j(x)}A_{ki}{}^{(x)}\\
&=\frac{c}{4}(p_{k}\phi_{ji}-p_{j}\phi_{ki}-2p_{i}\phi_{kj}+q_{k}\psi_{ji}-q_{j}\psi_{ki}-2q_{i}\psi_{kj}\\
&\quad+r_{k}\theta_{ji}-r_{j}\theta_{ki}-2r_{i}\theta_{kj}),\\
(2.27)\qquad &\nabla_{k}A_{ji}{}^{(x)}-\nabla_{j}A_{ki}{}^{(x)}+l_{k}{}^{(x)}A_{ji}-l_{j}{}^{(x)}A_{ki}+l_{k(y)}{}^{(x)}A_{ji}{}^{(y)}\\
&\quad-l_{j(y)}{}^{(x)}A_{ki}{}^{(y)}=0,
\end{aligned}
$$

and those of Ricci are given by

$$
\begin{aligned}
(2.28)\qquad &\nabla_{j}l_{i}{}^{(x)}-\nabla_{i}l_{j}{}^{(x)}+A_{j}{}^{h}A_{ih}{}^{(x)}-A_{i}{}^{h}A_{jh}{}^{(x)}\\
&\quad+l_{j(y)}{}^{(x)}l_{i}{}^{(y)}-l_{i(y)}{}^{(x)}l_{j}{}^{(y)}=0,\\
(2.29)\qquad &\nabla_{j}l_{i(x)}{}^{(y)}-\nabla_{i}l_{j(x)}{}^{(y)}+A_{j}{}^{h}{}_{(x)}A_{ih}{}^{(y)}-A_{i}{}^{h}{}_{(x)}A_{jh}{}^{(y)}\\
&+l_{j(x)}l_{i}{}^{(y)}-l_{i(x)}l_{j}{}^{(y)}+l_{j(z)}{}^{(y)}l_{i(x)}{}^{(z)}-l_{i(z)}{}^{(y)}l_{j(x)}{}^{(z)}\\
&=\frac{c}{2}(\phi_{ij}\phi_{(x)}{}^{(y)}+\psi_{ij}\psi_{(x)}{}^{(y)}+\theta_{ij}\theta_{(x)}{}^{(y)}).
\end{aligned}
$$

3. Submanifolds of a quaternionic Kaehlerian manifold admitting an almost contact metric compound 3-structure

In this section we asume that the ambient manifold $\widetilde{M}$ is a $4m$-dimensional quaternionic Kaehlerian manifold of a constant Q-sectional curvature c and that the metric compound 3-structure induced on an

$n(>1)$-dimensional submanifold M of $\widetilde{M}$ defines an almost contact metric 3-structure. Therefore $(\phi_j{}^h,\ \psi_j{}^h,\ \theta_j{}^h,\ g_{ji},\ p^h,\ q^h,\ r^h)$ defines an almost contact metric 3-structure on M.

We now assume that the second fundamental tensors and the structure tensors $\phi_j{}^h$, $\psi_j{}^h$ and $\theta_j{}^h$ commute each other, that is,

$$A_{jh}\phi_i{}^h+A_{ih}\phi_j{}^h=0,\quad A_{jh}{}^{(x)}\phi_i{}^h+A_{ih}{}^{(x)}\phi_j{}^h=0, \tag{3.1}$$
$$A_{jh}\psi_i{}^h+A_{ih}\psi_j{}^h=0,\quad A_{jh}{}^{(x)}\psi_i{}^h+A_{ih}{}^{(x)}\psi_j{}^h=0, \tag{3.2}$$
$$A_{jh}\theta_i{}^h+A_{ih}\theta_j{}^h=0,\quad A_{jh}{}^{(x)}\theta_i{}^h+A_{ih}{}^{(x)}\theta_j{}^h=0. \tag{3.3}$$

Transvecting the first equations of (3.1), (3.2), (3.3) with $\phi_k{}^i$, $\psi_k{}^i$, $\theta_k{}^i$ respectively and using (1.14), we obtain

$$A_{jh}p^h=\alpha p_j,\quad A_{jh}q^h=\alpha q_j,\quad A_{jh}r^h=\alpha r_j \tag{3.4}$$

with the aid of (1.14), (3.1), (3.2) and (3.3).

If we take the symmetric part of (2.20) in j and i and using (3.1) $\sim$ (3.3), then we obtain

$$\begin{aligned} 2A_{ji}{}^{(z)}&=-(l_j{}^{(y)}p_i+l_i{}^{(y)}p_j)\phi_{(y)}{}^{(z)},\\ 2A_{ji}{}^{(z)}&=-(l_j{}^{(y)}q_i+l_i{}^{(y)}q_j)\psi_{(y)}{}^{(z)},\\ 2A_{ji}{}^{(z)}&=-(l_j{}^{(y)}r_i+l_i{}^{(y)}r_j)\theta_{(y)}{}^{(z)}, \end{aligned} \tag{3.5}$$

from which, transvecting $p^i\phi_{(z)}{}^{(x)}$, $q^i\psi_{(z)}{}^{(x)}$, $r^i\theta_{(z)}{}^{(x)}$ respectively,

$$l_j{}^{(x)}=(l_h{}^{(x)}p^h)p_j=(l_h{}^{(x)}q^h)q_j=(l_h{}^{(x)}r^h)r_j \tag{3.6}$$

because of (2.9) $\sim$ (2.11) and (2.22). Therefore, (3.5) reduces to

$$A_{ji}{}^{(z)}=A^{(z)}p_jp_i=A^{(z)}q_jq_i=A^{(z)}r_jr_i \tag{3.7}$$

with the aid of (2.21). From (3.7), we have $A^{(z)}=0$. Using (2.22), we have the following lemma.

LEMMA 3.1. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a quaternionic Kaehlerian manifold $\widetilde{M}$ admitting an almost contact metric compound 3-structure. If the second fundamental tensors are commutative with the structure tensors ϕ, ψ and θ respectively induced on M, then we have*

$$A_{ji}{}^{(z)}=0,\quad l_j{}^{(z)}=0. \tag{3.8}$$

From (2.12), (2.13), (2.14) and (3.8), we have

PROPOSITION 3.2. *Under the same assumptions as those stated in Lemma 3.1, if the normal vectors* $C_{(x)}{}^A$ *are parallel in the subnormal bundle spanned by* $C_{(x)}{}^A$, *then* M *is contained as a hypersurface in* $(n+1)$-*dimensional quaternionic Kaehlerian manifold.*

From (2.29) and (3.8), we have

PROPOSITION 3.3. *Let* M *be an* $n(>1)$-*dimensional semi-invariant submanifold with the distinguished normal* C^A *of a quaternionic Kaehlerian manifold* $\widetilde{M}$ *of constant Q-sectional curvature c admitting an almost contact metric compound 3-structure. If the normal vectors* $C_{(x)}{}^A$ *are parallel in the subnormal bundle spanned by* $C_{(x)}{}^A$ *and if the second fundamental tensors are commutative with the structure tensors* ϕ, ψ *and* θ *respectively induced on* M, *then* M *is contained as a hypersurface in* $(n+1)$-*dimensional Euclidean space or* M *is a real hypersurface of quaternionic Kaehlerian manifold of constant Q-sectional curvature c.*

COROLLARY 3.4. *Under the same assumptions as those stated in Proposition 3.3, if the ambient manifold is a quaternionic projective space* QP^m, *then* M *is real hypersurface of* QP^m.

Differentiating (3.4) covariantly and then taking the skew symmetric part with respect to the indices k and j and using (2.16), (2.26) and (3.8),

$$(3.9)\qquad \begin{aligned}&\frac{c}{4}(2\phi_{jk}+2r_jq_k-2q_jr_k)+2A_j{}^tA_{th}\phi_k{}^h\\ &\quad=(\nabla_k\alpha)p_j-(\nabla_j\alpha)p_k+2\alpha A_{jh}\phi_k{}^h,\\ &\frac{c}{4}(2\psi_{jk}+2p_jr_k-2r_jp_k)+2A_j{}^tA_{th}\psi_k{}^h\\ &\quad=(\nabla_k\alpha)q_j-(\nabla_j\alpha)q_k+2\alpha A_{jh}\psi_k{}^h,\\ &\frac{c}{4}(2\theta_{jk}+2q_jp_k-2p_jq_k)+2A_j{}^tA_{th}\theta_k{}^h\\ &\quad=(\nabla_k\alpha)r_j-(\nabla_j\alpha)r_k+2\alpha A_{jh}\theta_k{}^h\end{aligned}$$

with the help of (1.14) and (3.1)~(3.3).

If we transvect (3.9) with p^j, q^j and r^j respectively and use (1.14) and (3.4), then we have

$$(3.10)\qquad \nabla_j\alpha=p^h(\nabla_h\alpha)p_j=q^h(\nabla_h\alpha)q_j=r^h(\nabla_h\alpha)r_j$$

from which, α is a constant. Thus, from (3.9), we have

$$A_{jh}A_i^h=\alpha A_{ji}+\frac{c}{4}(g_{ji}-p_jp_i-q_jq_i-r_jr_i)$$

with the aid of (1. 14) and (3. 4).

Thus we have

LEMMA 3. 5. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a quaternionic Kaehlerian manifold $\tilde{M}$ of constant Q-sectional curvature c admitting an almost contact metric compound 3-structure. If the second fundamental tensors of M are commutative with the structure tensors ϕ, ψ and θ respectively induced on M, then an eigen polynomial of the second fundamental tensor is given by*

$$A_{jh}A_i^h=\alpha A_{ji}+\frac{c}{4}(g_{ji}-p_jp_i-q_jq_i-r_jr_i), \tag{3.11}$$

where α is a constant.

From (3. 11), we have

$$A_{ji}A^{ji}=\alpha A_t^t+\frac{c}{4}(n-3) \tag{3.12}$$

from which, we can easily see that the square of the norm of A_{ji} is constant.

From (2. 26), we have $\nabla_kA_j^k=0$.
By using the Ricci identity of A_{ji}, we obtain

$$K_{jt}A_i^t-K_{kjih}A^{kh}=\nabla^k\nabla_jA_{ki} \tag{3.13}$$

If we make use of (2. 26) and (3. 8), then the above equation becomes

$$\begin{aligned}K_{jh}A_i^h-K_{kjih}A^{kh}=\nabla^k\{\nabla_kA_{ji}-\frac{c}{4}(&p_k\phi_{ji}-p_j\phi_{ki}\\&-2p_i\phi_{kj}+q_k\psi_{ji}-q_j\psi_{ki}-2q_i\psi_{kj}\\&+r_k\theta_{ji}-r_j\theta_{ki}-2r_i\theta_{kj})\}.\end{aligned} \tag{3.14}$$

Transvecting (3. 14) with A^{ji}, we have

$$\begin{aligned}A^{ji}(K_{jh}A_i^h-K_{kjih}A^{kh})=A^{ji}\nabla^k\nabla_kA_{ji}+\frac{3c}{4}A^{ji}\nabla^k(&p_j\phi_{ki}\\&+q_j\psi_{ki}+r_j\theta_{ki}).\end{aligned} \tag{3.15}$$

On the other hand, from (2. 25), we have

$$K_{ji}=\frac{c}{4}(n+7)g_{ji}+(A_t^t-\alpha)A_{ji}-\frac{c}{2}(p_jp_i+q_jq_i+r_jr_i). \tag{3.16}$$

Substituting (2.25) and (3.16) into (3.15) and using (2.15) and (2.16), (3.15) reduces to

$$A^{ji}\nabla^k\nabla_k A_{ji} = -\frac{3}{8}c^2(n-3). \tag{3.17}$$

Consider the following identity:

$$\frac{1}{2}\Delta(A_{ji}A^{ji}) = A^{ji}(\nabla^k\nabla_k A_{ji}) + \|\nabla_k A_{ji}\|^2, \tag{3.18}$$

where $\Delta = g^{ji}\nabla_j\nabla_i$.

Since $A^{ji}A_{ji}$ is constant and using (3.17), we obtain from (3.18)

$$\|\nabla_k A_{ji}\|^2 = \frac{3}{8}c^2(n-3). \tag{3.19}$$

Putting

$$\overset{*}{\nabla}_k A_{ji} = \nabla_k A_{ji} - \frac{c}{4}(\phi_{ik}p_j + \phi_{jk}p_i + \psi_{ik}q_j + \psi_{jk}q_i + \theta_{ik}r_j + \theta_{jk}r_i)$$

and using the equation (2.26), we can easily check that $\|\overset{*}{\nabla}_k A_{ij}\|^2 = 0$.

Thus we have

PROPOSITION 3.6. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a quaternionic Kaehlerian manifold $\widetilde{M}$ of constant Q-sectional curvature c admitting an almost contact metric compound 3-structure. If the second fundamental tensors are commutative with the structure tensors ϕ, ψ and θ respectively induced on M, then we have*

$$\|\nabla_k A_{ji}\|^2 = \frac{3}{8}c^2(n-3),$$

or equivalently

$$\nabla_k A_{ji} = \frac{c}{4}(\phi_{ik}p_j + \phi_{jk}p_i + \psi_{ik}q_j + \psi_{jk}q_i + \theta_{ik}r_j + \theta_{jk}r_i). \tag{3.20}$$

Furthermore, M is Euclidean if and only if the second fundamental tensor A_{ji} is parallel for $n>3$.

From (3.12) and Proposition 3.6, we have

THEOREM 3.7. *Let M be an $n(>1)$-dimensional complete semi-invariant submanifold with the distinguished normal C^A of a $4m$-dimensional Euclidean space admitting an almost contact metric compound 3-structure.*

If the second fundamental tensors of M are commutative with the structure tensors ϕ, ψ *and* θ *respectively induced on M, then M is an n-dimensional plane* E^n *or a product of a r-dimensional sphere and an* $(n-r)$*-dimensional plane* $S^r\times E^{n-r}$, where $0<r<n$.

4. Submanifolds of a quaternionic projective space

In this section we assume that the ambient manifold $\widetilde{M}$ is a quaternionic projective space QP^m, that is, $\widetilde{M}$ is a quaternionic Kaehlerian manifold of constant Q-sectional curvature 4, and assume that M is an $n(>1)$-dimensional submanifold of QP^m. As is well known, the unit sphere S^{4m+3} is a principal sphere bundle over a quaternionic projective space QP^m, which is characterized by the Hopf-fibration $\tilde{\pi}: S^{4m+3}\to QP^m$. We consider a Riemannian submersion $\pi:\overline{M}\to M$ compatible with $\tilde{\pi}: S^{4m+3}\to QP^m$, where $\tilde{\pi}^{-1}(M)=\overline{M}$. More precisely speaking, $\pi:\overline{M}\to M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\ \tilde{i}\ } & S^{4m+3} \\ {\scriptstyle\pi}\downarrow & \circlearrowright & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow[\ i\ ]{} & QP^m, \end{array}$$

where $\tilde{i}:\overline{M}\to S^{4m+3}$ and $i: M\to QP^m$ are certain isometric immersions. Let S^{4m+3} be covered by a system of coordinate neighborhoods $\{\hat{U}: y^\kappa\}$ such that $\tilde{\pi}(\hat{U})=\widetilde{U}$ are coordinate neighborhoods of QP^m with local coordinate systems (x^A). Then the projection $\tilde{\pi}: S^{4m+3}\to QP^m$ may be locally expressed by

$$x^A=x^A(y^\kappa) \tag{4.1}$$

and we put

$$E_\kappa{}^A=\partial_\kappa x^A,\quad \partial_\kappa=\partial/\partial y^\kappa, \tag{4.2}$$

where the matrix $(E_\kappa{}^A)$ has the maximal rank $4m$.

Let's denote by ξ^κ, η^κ and ζ^κ components of ξ, η and ζ of the induced Sasakian 3-structure $\{\xi, \eta, \zeta\}$ in S^{4m+3} respectively. Since ξ, η and ζ are the vertical vectors with respect to each fibre $\tilde{\pi}^{-1}(p)$ for every $p\in QP^m$,

$\{E_\kappa{}^A, \xi_\kappa, \eta_\kappa, \zeta_\kappa\}$ constitutes a local coframe in S^{4m+3}, where we have put $\xi_\kappa = g_{\kappa\mu}\xi^\mu$, $\eta_\kappa = g_{\kappa\mu}\eta^\mu$ and $\zeta_\kappa = g_{\kappa\mu}\zeta^\mu$ and $g_{\kappa\mu}$ denotes the fundamental metric tensors of S^{4m+3}. We denote by $\{E^\kappa{}_A, \xi^\kappa, \eta^\kappa, \zeta^\kappa\}$ the frame corresponding this coframe. Then we get

$$(4.3)\qquad \begin{aligned} &E_\kappa{}^A E^\kappa{}_B = \delta_B{}^A, \quad E_\kappa{}^A\xi^\kappa = 0, \quad E_\kappa{}^A\eta^\kappa = 0, \quad E_\kappa{}^A\zeta^\kappa = 0, \\ &\xi_\kappa E^\kappa{}_A = 0, \quad \eta_\kappa E^\kappa{}_A = 0, \quad \zeta_\kappa E^\kappa{}_A = 0. \end{aligned}$$

We now take coordinate neighborhoods $\{\bar{U} : y^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U}) = U$ are coordinate neighborhoods of M with local coordinate system (x^h). Let the isometric immersions $\tilde{i}$ and i are locally expressed by $y^\kappa = y^\kappa(y^\alpha)$ and $x^A = x^A(x^h)$ respectively. Then the commutativity $\tilde{\pi}\circ\tilde{i} = i\circ\pi$ of the preceding diagram implies

$$x^A(y^\kappa(y^\alpha)) = x^A(x^h(y^\alpha)),$$

where π is locally expressed by $x^h = x^h(y^\alpha)$. Which implies

$$(4.4)\qquad B_h{}^A E_\alpha{}^h = E_\kappa{}^A B_\alpha{}^\kappa,$$

where $B_\alpha{}^\kappa = \partial_\alpha y^\kappa$ and $E_\alpha{}^h = \partial_\alpha x^h$.

For each point $p\in M$, we can choose the mutually orthogonal unit normal vector fields $C_x{}^A$ defined in a neighborhood U of p such that $\{B_i{}^A,\ C_x{}^A\}$ generates the tangent space of QP^m at $i(p)$. Let $\bar{p}$ be an arbitrary point of the fibre $\pi^{-1}(p)$ over p, then the horizontal lifts $C_x{}^\kappa$ of $C_x{}^A$ are mutually orthogonal unit normal to $\bar{M}$ defined in the tubular neighborhood of $\bar{p}$ over U because of (4.4).

Since any fibre $\tilde{\pi}^{-1}(\tilde{p})$, $\tilde{p}\in QP^m$, is a maximal integral manifold of the distribution spanned by ξ, η and ζ, ξ^κ, η^κ and ζ^κ can be represented by

$$(4.5)\qquad \xi^\kappa = \xi^\alpha B_\alpha{}^\kappa, \quad \eta^\kappa = \eta^\alpha B_\alpha{}^\kappa, \quad \zeta^\kappa = \zeta^\alpha B_\alpha{}^\kappa,$$

$$(4.6)\qquad \xi^\alpha E_\alpha{}^h = 0, \quad \eta^\alpha E_\alpha{}^h = 0, \quad \zeta^\alpha E_\alpha{}^h = 0,$$

where ξ^α, η^α and ζ^α are vector fields in $\bar{M}$ which are vertical and span the tangent space to the fibre $\mathcal{F}$ at each point of $\bar{M}$ because of (4.3) and (4.4). Then (4.5) implies

$$(4.7)\qquad \xi_\alpha\xi^\alpha = 1, \quad \eta^\alpha\eta_\alpha = 1, \quad \zeta_\alpha\zeta^\alpha = 1,$$

because of $\xi_\kappa\xi^\kappa = \eta_\kappa\eta^\kappa = \zeta_\alpha\zeta^\alpha = 1$, where $\xi_\alpha = \xi^\beta g_{\beta\alpha}$, $\eta_\alpha = \eta^\beta g_{\beta\alpha}$ and $\zeta_\alpha = \zeta^\beta g_{\beta\alpha}$, $g_{\beta\alpha}$ being the fundamental metric tensor of $\bar{M}$ induced from $g_{\kappa\mu}$ in such a way that $g_{\beta\alpha}=g_{\kappa\mu}B_\beta{}^\kappa B_\alpha{}^\mu$. Therefore, $\{E_\alpha{}^h, \xi_\alpha, \eta_\alpha, \zeta_\alpha\}$ forms a local coframe in $\overline{M}$ corresponding $\{E_\kappa{}^A, \xi_\kappa, \eta_\kappa, \zeta_\kappa\}$ in S^{4m+3}. Denoting by $\{E^\alpha{}_h, \zeta^\alpha, \eta^\alpha, \zeta^\alpha\}$ the frame corresponding this coframe, we have

$$(4.8)\qquad \begin{aligned} &E^h{}_\alpha E^\alpha{}_k=\delta_k{}^h,\ \ \xi_\alpha E^\alpha{}_h=0,\ \ \eta_\alpha E^\alpha{}_h=0,\ \ \zeta_\alpha E^\alpha{}_h=0,\\ &\xi^\alpha E_\alpha{}^h=0,\ \ \eta^\alpha E_\alpha{}^h=0,\ \ \zeta^\alpha E_\alpha{}^h=0. \end{aligned}$$

We now put in $\hat{U}$

$$\xi=a^sC_s,\ \ \eta=b^sC_s,\ \ \zeta=c^sC_s,\ \ a_s=a^tg_{ts},\ \ b_s=b^tg_{ts},\ \ c_s=c^tg_{ts},$$

where $g_{ts}=g_{\lambda\mu}C^\lambda{}_tC^\mu{}_s$ and $g_{\lambda\mu}$ components of the induced metric in S^{4m+3} $(\subset Q^{m+1})$. Then it follows that

$$(4.9)\qquad C_s=a_s\xi+b_s\eta+c_s\zeta,\ \ a_sa^t+b_sb^t+c_sc^t=\delta_s^t.$$

Transvecting (4.4) with $E^\mu{}_A$ and substituting (4.9) imlpy

$$(E^\mu{}_AB_h{}^A)E_\alpha{}^h=B_\alpha{}^\mu-(a^s\xi_\alpha+b^s\eta_\alpha+c^s\zeta_\alpha)C_s{}^\mu,$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$, $\eta_\alpha=\eta^\beta g_{\beta\alpha}$ and $\zeta_\alpha=\zeta^\beta g_{\beta\alpha}$. Thus, transvecting the equation above with $E^\alpha{}_j$ and using (4.8), we have

$$(4.10)\qquad E^\kappa{}_AB_h{}^A=B_\alpha{}^\kappa E^\alpha{}_h.$$

Denoting by $\{{}_\mu{}^\lambda{}_\nu\}$, $\{{}_B{}^A{}_C\}$, $\{{}_\beta{}^\alpha{}_\gamma\}$, and $\{{}_j{}^h{}_i\}$ are Christoffel symbols formed with the Riemannian metrics $g_{\lambda\mu}$, g_{BA}, $g_{\alpha\beta}$ and g_{ji} respectively, we put

$$\begin{aligned} &D_\mu E_\lambda{}^A=\partial_\mu E_\lambda{}^A-\{{}_\mu{}^\kappa{}_\lambda\}E_\kappa{}^A+\{{}_B{}^A{}_C\}E_\mu{}^BE_\lambda{}^C,\\ &D_\mu E^\lambda{}_A=\partial_\mu E^\lambda{}_A+\{{}_\mu{}^\lambda{}_\kappa\}E_A{}^\kappa-\{{}_A{}^B{}_C\}E_\mu{}^CE^\lambda{}_B,\\ &\overline{\nabla}_\beta E_\alpha{}^h=\partial_\beta E_\alpha{}^h-\{{}_\beta{}^\gamma{}_\alpha\}E_\gamma{}^h+\{{}_j{}^h{}_i\}E_\beta{}^jE_\alpha{}^i,\\ &\overline{\nabla}_\beta E^\alpha{}_h=\partial_\beta E^\alpha{}_h+\{{}_\beta{}^\alpha{}_\gamma\}E^\gamma{}_h-\{{}_j{}^i{}_h\}E_\beta{}^jE^i{}_\alpha. \end{aligned}$$

Since the metrics $g_{\lambda\mu}$ and $g_{\alpha\beta}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of $E_\lambda{}^A$, $E^\lambda{}_A$, $E_\alpha{}^h$ and $E^\alpha{}_h$ are given by

$$(4.11)\qquad D_\mu E_\lambda{}^A=h_B{}^A{}_s(E_\mu{}^BC_\lambda{}^s+C_\mu{}^sE_\lambda{}^B),\ \ D_\mu E^\lambda{}_A=h_{BA}{}^sE_\mu{}^BC^\lambda{}_s-h_A{}^B{}_sC_\mu{}^sE^\lambda{}_B,$$

$$(4.12)\qquad \overline{\nabla}_\beta E_\alpha{}^h=h_j{}^h{}_s(E_\beta{}^jC_\alpha{}^s+C_\beta{}^sE_\alpha{}^j),\ \ \overline{\nabla}_\beta E^\alpha{}_h=h_{jk}{}^sE_\beta{}^jC^\alpha{}_s-h_k{}^j{}_sC_\beta{}^sF^\alpha{}_j$$

respectively, where D_μ and $\overline{\nabla}_\beta$ are the operator of the covariant differentiation of S^{4m+3} and $\overline{M}$ respectively, $h_B{}^A{}_s=g^{AC}g_{st}h_{BC}{}^t$, $h_j{}^h{}_s=g^{hi}g_{st}h_{ji}{}^t$, $h_{BA}{}^s$

and $h_{ji}{}^s$ are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively ([7]).

On the other hand, the equations of Gauss for $\overline{M}$ are given by

$$\bar{\nabla}_\beta B_\alpha{}^\kappa = A_{\beta\alpha} C^\kappa + A_{\beta\alpha}{}^{(x)} C_{(x)}{}^\kappa, \tag{4.13}$$

where $A_{\beta\alpha}$ and $A_{\beta\alpha}{}^{(x)}$ are the second fundamental tensors with respect to the normals $C^\kappa = C^A E^\kappa{}_A$ and $C_{(x)}{}^\kappa = C_{(x)}{}^A E^\kappa{}_A$ respectively, and those of Weingarten by

$$\bar{\nabla}_\beta C^\kappa = -A_\beta{}^\alpha B_\alpha{}^\kappa + l_\beta{}^{(x)} C_{(x)}{}^\kappa, \tag{4.14}$$

$$\bar{\nabla}_\beta C_{(x)}{}^\kappa = -A_\beta{}^\alpha{}_{(x)} B_\alpha{}^\kappa - l_{\beta(x)} C^\kappa + l_{\beta(x)}{}^{(y)} C_{(y)}{}^\kappa, \tag{4.15}$$

where $A_\beta{}^\alpha = g^{\gamma\alpha} A_{\beta\gamma}$, $A_\beta{}^\alpha{}_{(x)} = g^{\gamma\alpha} g_{(y)(x)} A_{\beta\gamma}{}^{(y)} = g^{\alpha\gamma} A_{\beta\gamma(x)}$, $l_\beta{}^{(x)}$ and $l_{\beta(x)}{}^{(y)}$ the third fundamental tensor and $l_{\beta(x)} = l_\beta{}^{(y)} g_{(y)(x)}$. Moreover in this case (4.4) and (4.10) imply $\nabla_j = E^\alpha{}_j \bar{\nabla}_\alpha$. Putting $\tilde{\phi}_\mu{}^\lambda = D_\mu \xi^\lambda$, $\tilde{\psi}_\mu{}^\lambda = D_\mu \eta^\lambda$ and $\tilde{\theta}_\mu{}^\lambda = D_\mu \zeta^\lambda$, we have by definition of Sasakian 3-structure

$$\begin{aligned}
&\tilde{\phi}_\mu{}^\lambda \tilde{\phi}_\kappa{}^\mu = -\delta_\kappa{}^\lambda + \xi_\kappa \xi^\lambda, \quad \tilde{\phi}_\mu{}^\lambda \xi^\mu = 0, \quad \xi_\lambda \tilde{\phi}_\mu{}^\lambda = 0, \quad \xi_\lambda \xi^\lambda = 1, \\
&\tilde{\psi}_\mu{}^\lambda \tilde{\psi}_\kappa{}^\mu = -\delta_\kappa{}^\lambda + \eta_\kappa \eta^\lambda, \quad \tilde{\psi}_\mu{}^\lambda \eta^\mu = 0, \quad \eta_\lambda \tilde{\psi}_\mu{}^\lambda = 0, \quad \eta_\lambda \eta^\lambda = 1, \\
&\tilde{\theta}_\mu{}^\lambda \tilde{\theta}_\kappa{}^\mu = -\delta_\kappa{}^\lambda + \zeta_\kappa \zeta^\lambda, \quad \tilde{\theta}_\mu{}^\lambda \zeta^\mu = 0, \quad \zeta_\lambda \tilde{\theta}_\mu{}^\lambda = 0, \quad \zeta_\lambda \zeta^\lambda = 1, \\
&\tilde{\phi}_\mu{}^\lambda \tilde{\psi}_\kappa{}^\mu = \tilde{\theta}_\kappa{}^\lambda + \eta_\kappa \xi^\lambda, \quad \tilde{\psi}_\mu{}^\lambda \tilde{\phi}_\kappa{}^\mu = -\tilde{\theta}_\kappa{}^\lambda + \xi_\kappa \eta^\lambda, \\
&\tilde{\psi}_\mu{}^\lambda \tilde{\theta}_\kappa{}^\mu = \tilde{\phi}_\kappa{}^\lambda + \zeta_\kappa \eta^\lambda, \quad \tilde{\theta}_\mu{}^\lambda \tilde{\psi}_\kappa{}^\mu = -\tilde{\phi}_\kappa{}^\lambda + \eta_\kappa \zeta^\lambda, \\
&\tilde{\theta}_\mu{}^\lambda \tilde{\phi}_\kappa{}^\mu = \tilde{\psi}_\kappa{}^\lambda + \xi_\kappa \zeta^\lambda, \quad \tilde{\phi}_\mu{}^\lambda \tilde{\psi}_\kappa{}^\mu = -\tilde{\psi}_\kappa{}^\lambda + \zeta_\kappa \xi^\lambda, \\
&\tilde{\phi}_\mu{}^\lambda \eta^\mu = \zeta^\lambda, \quad \tilde{\psi}_\mu{}^\lambda \zeta^\mu = -\eta^\lambda, \quad \tilde{\psi}_\mu{}^\lambda \zeta^\mu = \xi^\lambda, \\
&\tilde{\theta}_\mu{}^\lambda \xi^\mu = -\zeta^\lambda, \quad \tilde{\theta}_\mu{}^\lambda \xi^\mu = \eta^\lambda, \quad \tilde{\phi}_\mu{}^\lambda \eta^\mu = -\xi^\lambda, \\
&\tilde{\phi}_{\mu\lambda} = -\tilde{\psi}_{\lambda\mu}, \quad \tilde{\psi}_{\mu\lambda} = -\tilde{\psi}_{\lambda\mu}, \quad \tilde{\theta}_{\mu\lambda} = -\tilde{\theta}_{\lambda\mu},
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
D_\mu \tilde{\phi}_\lambda{}^\kappa = \xi_\lambda \delta_\mu{}^\kappa - \xi^\kappa g_{\mu\lambda}, \quad D_\mu \tilde{\phi}_\lambda{}^\kappa = \eta^\lambda \delta_\mu{}^\kappa - \eta^\kappa g_{\mu\lambda}, \\
D_\mu \tilde{\theta}_\lambda{}^\kappa = \zeta_\lambda \delta_\mu{}^\kappa - \zeta^\kappa g_{\mu\lambda},
\end{aligned} \tag{4.17}$$

where we have put $\xi_\kappa = \xi^\lambda g_{\lambda\kappa}$, $\eta_\kappa = \eta^\lambda g_{\lambda\kappa}$, $\zeta_\kappa = \zeta^\lambda g_{\lambda\kappa}$, $\tilde{\phi}_{\mu\lambda} = \tilde{\psi}_\mu{}^\nu g_{\nu\lambda}$, $\tilde{\psi}_{\mu\lambda} = \tilde{\phi}_\mu{}^\nu g_{\nu\lambda}$ and $\tilde{\theta}_{\mu\lambda} = \tilde{\theta}_\mu{}^\nu g_\nu{}^\lambda$ ([10]). Denoting by L_ξ, L_η and L_ζ the Lie differentiation with respect to ξ, η and ζ respectively, we find

$$\begin{aligned}
&L_\xi \tilde{\psi}_\lambda{}^\mu = 0, \quad L_\eta \tilde{\psi}_\lambda{}^\mu = 2\tilde{\theta}_\lambda{}^\mu, \quad L_\zeta \tilde{\phi}_\lambda{}^\mu = -2\tilde{\phi}_\lambda{}^\kappa, \\
&L_\xi \tilde{\phi}_\lambda{}^\mu = -2\tilde{\theta}_\lambda{}^\mu, \quad L_\eta \tilde{\phi}_\lambda{}^\mu = 0, \quad L_\zeta \tilde{\psi}_\lambda{}^\mu = 2\tilde{\phi}_\lambda{}^\mu, \\
&L_\xi \tilde{\theta}_\lambda{}^\mu = 2\tilde{\psi}_\lambda{}^\mu, \quad L_\eta \tilde{\theta}_\lambda{}^\mu = -2\tilde{\phi}_\lambda{}^\mu, \quad L_\zeta \tilde{\theta}_\lambda{}^\lambda = 0
\end{aligned} \tag{4.18}$$

because of $D_\lambda\xi^\mu=\tilde{\phi}_\lambda{}^\mu$, $D_\lambda\eta^\mu=\tilde{\psi}_\lambda{}^\mu$, $D_\lambda\zeta^\mu=\tilde{\theta}_\lambda{}^\mu$ and (4.18).

Putting

$$\phi_B{}^A=\tilde{\phi}_\mu{}^\lambda E^\mu{}_B E_\lambda{}^A,\quad \psi_B{}^A=\tilde{\psi}_\mu{}^\lambda E^\mu{}_B E_\lambda{}^A,\quad \theta_B{}^A=\tilde{\theta}_\mu{}^\lambda E^\mu{}_B E_\lambda{}^A, \tag{4.19}$$

we can see that $\phi_B{}^A$, $\psi_B{}^A$ and $\theta_B{}^A$ defines a global tensor fields of the same type as that of $\tilde{\psi}_\mu{}^\lambda$, $\tilde{\phi}_\mu{}^\lambda$ and $\tilde{\theta}_\mu{}^\lambda$ because of (4.18), $LE^\lambda{}_A=0$ and $LE_\lambda{}^A=0$ ([7]).

Differentiating $\xi^\varepsilon E_\varepsilon{}^A=0$, $\eta^\varepsilon E_\varepsilon{}^A=0$ and $\zeta^\varepsilon E_\varepsilon{}^A=0$ covariantly along S^{4m+3} and using (4.9), (4.11), (4.17) and (4.19), we find

$$h_{BA}{}^s=-(a^s\phi_{BA}+b^s\psi_{BA}+c^s\theta_{BA}), \tag{4.20}$$

where $\phi_{BA}=\phi_B{}^C g_{CA}$, $\psi_{BA}=\psi_B{}^C g_{CA}$ and $\theta_{BA}=\theta_B{}^C g_{CA}$.

We also have by using (4.3), (4.9), (4.16) and (4.19)

$$\begin{aligned} &\phi_B{}^A\phi_C{}^B=-\delta_C{}^A,\quad \psi_B{}^A\psi_C{}^B=-\delta_C{}^A,\quad \theta_B{}^A\theta_C{}^B=-\delta_C{}^A,\\ &\phi_B{}^A\psi_C{}^B=-\psi_B{}^A\phi_C{}^B=\theta_C{}^A,\quad \psi_B{}^A\theta_C{}^B=-\theta_B{}^A\psi_C{}^B=\phi_C{}^A,\\ &\theta_B{}^A\phi_C{}^B=-\phi_B{}^A\theta_C{}^B=\psi_C{}^A. \end{aligned} \tag{4.21}$$

Consider a point $\tilde{p}$ of QP^m and a point $\hat{p}$ of S^{4m+3} such that $\tilde{\pi}(\hat{p})=\tilde{p}$. Denoting by $\tilde{\phi}_{\hat{p}}, \tilde{\psi}_{\hat{p}}$ and $\tilde{\theta}_{\hat{p}}$ respectively the values of $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\theta}$ at $\hat{p}$, we can define tensor $\tilde{F}_{\hat{p}}, \tilde{G}_{\hat{p}}$ and $\tilde{H}_{\hat{p}}$ of type (1.1) at $\tilde{p}\in QP^m$ respectively by

$$\tilde{F}_{\hat{p}}A=d\tilde{\pi}(\tilde{\phi}_{\hat{p}}A^L),\quad \tilde{G}_{\hat{p}}A=d\tilde{\pi}(\tilde{\psi}_{\hat{p}}A^L),\quad \tilde{H}_{\hat{p}}A=d\tilde{\pi}(\tilde{\theta}_{\hat{p}}A^L) \tag{4.22}$$

for any vector A tangent to QP^m at $\tilde{p}$, where $d\tilde{\pi}$ means the differential of $\tilde{\pi}$ and A^L denotes the horizontal lift of A. We now denote by $V_{\tilde{p}}{}^*$ the lenear closure of the set

$$\Big(\bigcup_{\hat{p}\in\tilde{\pi}^{-1}(\tilde{p})}\tilde{F}_{\hat{p}}\Big)\cup\Big(\bigcup_{\hat{p}\in\tilde{\pi}^{-1}(\tilde{p})}\tilde{G}_{\hat{p}}\Big)\cup\Big(\bigcup_{\hat{p}\in\tilde{\pi}^{-1}(\tilde{p})}\tilde{H}_{\hat{p}}\Big)$$

of tensors of type (1.1) at $\tilde{p}\in QP^m$ and $V_{\tilde{p}}=\bigcup_{\tilde{p}\in QP^m} V^{\tilde{p}*}$, which is a linear subbundle of the tensor bundle of type (1.1) over QP^m.

Take a coordinate neighborhood $\tilde{U}$ at $\tilde{p}\in QP^m$ and consider a local cross-section τ of S^{4m+3} over $\tilde{U}$. If we put

$$F_{\tilde{p}}=\tilde{F}_{\tau(\tilde{p})},\quad G_{\tilde{p}}=\tilde{G}_{\tau(\tilde{p})},\quad H_{\tilde{p}}=\tilde{H}_{\tau(\tilde{p})},\quad \tilde{p}\in\tilde{U}, \tag{4.23}$$

then the correspondence $\tilde{p}\to F_{\tilde{p}}$, $\tilde{p}\to G_{\tilde{p}}$, and $\tilde{p}\to H_{\tilde{p}}$ define respectively local tensor fields F, G and H of type (1.1) on $\tilde{U}$. Thus, taking account of (4.21), (4.22) and (4.23), we find

$$(4.24)\quad \begin{aligned} &F_B{}^A F_C{}^B = -\delta_C{}^A,\ G_B{}^A G_C{}^B = -\delta_C{}^A,\ H_B{}^A H_C{}^B = -\delta_C{}^A,\\ &F_B{}^A G_C{}^B = -G_B{}^A F_C{}^B = H_C{}^A,\ G_B{}^A H_C{}^B = -H_B{}^A G_C{}^B = F_C{}^A,\\ &H_B{}^A F_C{}^B = -F_B{}^A H_C{}^B = G_C{}^A,\ F_{BA} = -F_{AB},\ G_{BA} = -G_{AB},\\ &H_{BA} = -H_{AB}, \end{aligned}$$

where $F_{BA}=F_B{}^C g_{AC}$, $G_{BA}=G_B{}^C g_{AC}$, $H_{BA}=H_B{}^C g_{AC}$, $F_B{}^A$, $G_B{}^A$ and $H_B{}^A$ being respectively local components of F, G and H in U.

Next, denoting by $(\tau^\kappa(x))$ coordinates of the point $\tau(\tilde{p})$, we have from (4.23)

$$F_B{}^A(x) = \phi_B{}^A(\tau^\kappa(x)),\quad G_B{}^A(x) = \psi_B{}^A(\tau^\kappa(x)),\quad H_B{}^A(x) = \theta_B{}^A(\tau^\kappa(x)).$$

Differentiating the first equation above with respect to x^C and using $(\partial_C\tau^\kappa)E_\kappa{}^A=\delta_C{}^A$ implies $\partial_C F_B{}^A = \partial_C\phi_B{}^A + (\partial_C\tau^\kappa)C_\kappa{}^s\partial_s\phi_B{}^A$. Thus, taking account of (4.20), we have $\tilde{\nabla}_C F_B{}^A = w_C G_B{}^A - v_C H_B{}^A$, where we have put $v_C = -b_s C_\kappa{}^s\partial_C\tau^\kappa$ and $w_C = -c_s C_\kappa{}^s\partial_C\tau^\kappa$. Similarly, using (4.20), we obtain in $\tilde{U}$

$$(4.25)\quad \begin{aligned} &\tilde{\nabla}_C F_B{}^A = w_C G_B{}^A - v_C H_B{}^A,\ \tilde{\nabla}_C G_B{}^A = -w_C F_B{}^A + u_C H_B{}^A,\\ &\tilde{\nabla}_C H_B{}^A = v_C F_B{}^A - u_C G_B{}^A \end{aligned}$$

for certain local 1-forms u, v, w defined in $\tilde{U}$. By means of (4.24) and (4.25) the quaternionic projective space QP^m admits a quaternionic Kaehlerian structure ([4], [5], [7]).

On the other hand, by taking account of the co-Gauss equation for the submersion $\tilde{\pi}: S^{4m+3}\to QP^m$ and (4.20), we can see that the base space QP^m is a quaternionic Kaehlerian manifold of constant Q-sectional curvature 4 given by (2.24).

As to transforms of $B_\alpha{}^\mu$ and $C_x{}^\lambda$ by $\tilde{\phi}_\mu{}^\kappa, \tilde{\psi}_\mu{}^\kappa$ and $\tilde{\theta}_\mu{}^\kappa$, we have

$$\begin{aligned} &(4.26)\quad \tilde{\phi}_\mu{}^\kappa B_\alpha{}^\mu = \phi_\alpha{}^\beta B_\beta{}^\kappa + \phi_\alpha{}^x C_x{}^\kappa,\quad \tilde{\phi}_\mu{}^\kappa C_x{}^\mu = -\phi_x{}^\beta B_\beta{}^\kappa + {}'\phi_x{}^y C_y{}^\kappa,\\ &(4.27)\quad \tilde{\psi}_\mu{}^\kappa B_\alpha{}^\mu = \psi_\alpha{}^\beta B_\beta{}^\kappa + \psi_\alpha{}^x C_x{}^\kappa,\quad \tilde{\psi}_\mu{}^\kappa C_x{}^\mu = -\psi_x{}^\beta B_\beta{}^\kappa + {}'\psi_x{}^y C_y{}^\kappa,\\ &(4.28)\quad \tilde{\theta}_\mu{}^\kappa B_\alpha{}^\mu = \theta_\alpha{}^\beta B_\beta{}^\kappa + \theta_\alpha{}^x C_x{}^\kappa,\quad \tilde{\theta}_\mu{}^\kappa C_x{}^\mu = -\theta_x{}^\beta B_\beta{}^\kappa + {}'\theta_x{}^y C_y{}^\kappa, \end{aligned}$$

where $\phi_\alpha{}^\beta$, $\psi_\alpha{}^\beta$ and $\theta_\alpha{}^\beta$ are tensor fields of type (1.1), $\phi_\alpha{}^x$, $\psi_\alpha{}^x$ and $\theta_\alpha{}^x$ 1-forms for fixed x, $\phi_x{}^\beta$, $\psi_x{}^\beta$ and $\theta_x{}^\beta$ vector fields associated with $\phi_\beta{}^x$, $\psi_\beta{}^x$ and $\theta_\beta{}^x$ defined by $\phi_\beta{}^x = \phi_y{}^\alpha g_{\alpha\beta} g^{yx}$, $\psi_\beta{}^x = \psi_y{}^\alpha g_{\alpha\beta} g^{yx}$ and $\theta_\beta{}^x = \theta_y{}^\alpha g_{\alpha\beta} g^{yx}$ and ${}'\phi_x{}^y$, ${}'\psi_x{}^y$ and ${}'\theta_x{}^y$ scalar fields for fixed x and y on $\overline{M}$.

Now we suppose that n-dimensional submanifold M of QP^m is semi-invariant with respect to the distinguished normal C^A. Then we can have the algebraic relationships (1. 14) and (2. 9)~(2. 11) and the structure equations (2. 15)~(2. 19)

If we make use of (1. 5)~(1. 7), (4. 4), (4. 10), (4. 19) and (4. 26)~(4. 28), then we have

$$(4.29)\quad \begin{aligned} &\phi_j{}^h=\phi^\alpha{}_\beta E_j{}^\beta E_\alpha{}^h,\quad \psi_j{}^h=\psi_\beta{}^\alpha E^\beta{}_j E_\alpha{}^h,\quad \theta_j{}^h=\theta_\beta{}^\alpha E^\beta{}_j E_\alpha{}^h,\\ &p_j=\phi_\beta{}^{1*}E^\beta{}_j,\quad q_j=\psi_\beta{}^{1*}E^\beta{}_j,\quad r_j=\theta_\beta{}^{1*}E^\beta{}_j\\ &p^i=\phi_{1*}{}^\beta E_\beta{}^i,\quad q^i=\psi_{1*}{}^\beta E_\beta{}^i,\quad r^i=\theta_{1*}{}^\beta E_\beta{}^i,\\ &{}'\phi_{(x)}{}^{(y)}=\phi_{(x)}{}^{(y)},\quad {}'\psi_{(x)}{}^{(y)}=\psi_{(x)}{}^{(y)},\quad {}'\theta_{(x)}{}^{(y)}=\theta_{(x)}{}^{(y)}.\end{aligned}$$

Thus, from (4. 26), (4. 27) and (4. 28), we have

$$(4.30)\quad \tilde\phi_\mu{}^\epsilon B_\alpha{}^\mu=\phi_\alpha{}^\beta B_\beta{}^\epsilon+p_\alpha C^\epsilon,\quad \tilde\phi_\mu{}^\epsilon C^\mu=-p^\alpha B_\alpha{}^\epsilon,\quad \tilde\phi_\mu{}^\epsilon C_{(x)}{}^\mu=\phi_{(x)}{}^{(y)}C_{(y)}{}^\epsilon,$$
$$(4.31)\quad \tilde\psi_\mu{}^\epsilon B_\alpha{}^\mu=\psi_\alpha{}^\beta B_\beta{}^\epsilon+q_\alpha C^\epsilon,\quad \tilde\psi_\mu{}^\epsilon C^\mu=-q^\alpha B_\alpha{}^\epsilon,\quad \tilde\psi_\mu{}^\epsilon C_{(x)}{}^\mu=\psi_{(x)}{}^{(y)}C_{(y)}{}^\epsilon,$$
$$(4.32)\quad \tilde\theta_\mu{}^\epsilon B_\alpha{}^\mu=\theta_\alpha{}^\beta B_\beta{}^\epsilon+r_\alpha C^\epsilon,\quad \tilde\theta_\mu{}^\epsilon C^\mu=-r^\alpha B_\alpha{}^\epsilon,\quad \tilde\theta_\mu{}^\epsilon C_{(x)}{}^\mu=\theta_{(x)}{}^{(y)}C_{(y)}{}^\epsilon,$$

where we have put $\phi_\beta{}^{1*}=p_\beta$, $\psi_\beta{}^{1*}=q_\beta$, $\theta_\beta{}^{1*}=r_\beta$, $\phi_{1*}{}^\beta=p^\beta$, $\psi_{1*}{}^\beta=q^\beta$ and $\theta_{1*}{}^\beta=r^\beta$.

Applying $\tilde\phi, \tilde\psi$ and $\tilde\theta$ to (4. 30), (4. 31) and (4. 32) respectively and using (4. 16), we get

$$(4.33)\quad \begin{aligned} &\phi_\gamma{}^\alpha\phi_\beta{}^\gamma=-\delta_\beta{}^\alpha+p_\beta p^\alpha+\xi_\beta\xi^\alpha,\quad \phi_{\beta\alpha}p^\beta=\phi_\beta{}^\alpha\xi^\beta=0,\quad p_\alpha p^\alpha=1,\\ &\xi_\alpha\xi^\alpha=1,\quad \psi_\gamma{}^\alpha\psi_\beta{}^\gamma=-\delta_\beta{}^\alpha+q_\beta q^\alpha+\eta_\beta\eta^\alpha,\quad \psi_\beta{}^\alpha q^\beta=\psi_\beta{}^\alpha\eta^\beta=0,\\ &q_\alpha q^\alpha=1,\quad \eta_\alpha\eta^\alpha=1,\quad \theta_\gamma{}^\alpha\theta_\beta{}^\gamma=-\delta_\beta{}^\alpha+r_\beta r^\alpha+\zeta_\beta\zeta^\alpha,\\ &\theta_\beta{}^\alpha r^\beta=\theta_\beta{}^\alpha\zeta^\beta=0,\quad r_\alpha r^\alpha=1,\quad \zeta_\alpha\zeta^\alpha=1,\\ &\phi_\gamma{}^\alpha\psi_\beta{}^\gamma=\theta_\beta{}^\alpha+q_\beta p^\alpha+\eta_\beta\xi^\alpha,\quad \psi_\beta{}^\alpha p_\alpha=r_\beta,\quad p_\alpha\xi^\alpha=0,\\ &\psi_\gamma{}^\alpha\phi_\beta{}^\gamma=-\theta_\beta{}^\alpha+p_\beta q^\alpha+\xi_\beta\eta^\alpha,\quad \phi_\beta{}^\alpha q_\alpha=-r_\beta,\quad q_\alpha\eta^\alpha=0,\\ &\psi_\gamma{}^\alpha\theta_\beta{}^\gamma=\phi_\beta{}^\alpha+r_\beta q^\alpha+\zeta_\beta\eta^\alpha,\quad \theta_\beta{}^\alpha q_\alpha=p_\beta,\quad r_\alpha\zeta^\alpha=0,\\ &\theta_\gamma{}^\alpha\psi_\beta{}^\gamma=-\phi_\beta{}^\alpha+q_\beta r^\alpha+\eta_\beta\zeta^\alpha,\quad \psi_\beta{}^\alpha r_\alpha=-p_\beta,\quad \xi_\alpha p^\alpha=0,\\ &\theta_\gamma{}^\alpha\phi_\beta{}^\gamma=\psi_\beta{}^\alpha+p_\beta r^\alpha+\xi_\beta\zeta^\alpha,\quad \phi_\beta{}^\alpha r_\alpha=q_\beta,\quad \eta_\alpha q^\alpha=0,\\ &\phi_\gamma{}^\alpha\theta_\beta{}^\gamma=-\psi_\beta{}^\alpha+r_\beta p^\alpha+\zeta_\beta\xi^\alpha,\quad \theta_\beta{}^\alpha p_\alpha=-q_\beta,\quad \zeta_\alpha r^\alpha=0,\\ &\theta_\beta{}^\alpha\eta^\beta=-\psi_\beta{}^\alpha\zeta^\beta=\xi^\alpha,\quad r_\alpha\eta^\alpha=0,\quad q_\alpha\zeta^\alpha=0,\quad \xi_\alpha\eta^\alpha=0,\\ &\phi_\beta{}^\alpha\zeta^\beta=-\theta_\beta{}^\alpha\xi^\beta=\eta^\alpha,\quad p_\alpha\zeta^\alpha=0,\quad r_\alpha\xi^\alpha=0,\quad \eta_\alpha\zeta^\alpha=0,\\ &\psi_\beta{}^\alpha\xi^\beta=-\phi_\beta{}^\alpha\eta^\beta=\zeta^\alpha,\quad p_\alpha\eta^\alpha=0,\quad q_\alpha\xi^\alpha=0,\quad \zeta_\alpha\eta^\alpha=0,\\ &\phi_{(x)}{}^{(y)}\phi_{(y)}{}^{(z)}=-\delta_{(x)}{}^{(z)},\quad \psi_{(x)}{}^{(y)}\psi_{(y)}{}^{(z)}=-\delta_{(x)}{}^{(z)},\\ &\theta_{(x)}{}^{(y)}\theta_{(y)}{}^{(z)}=-\delta_{(x)}{}^{(z)},\\ &\psi_{(x)}{}^{(y)}\phi_{(y)}{}^{(z)}=-\phi_{(x)}{}^{(y)}\psi_{(y)}{}^{(z)}=\theta_{(x)}{}^{(z)},\\ &\theta_{(x)}{}^{(y)}\psi_{(y)}{}^{(z)}=-\psi_{(x)}{}^{(y)}\theta_{(y)}{}^{(z)}=\phi_{(x)}{}^{(z)},\\ &\phi_{(x)}{}^{(y)}\theta_{(y)}{}^{(z)}=-\theta_{(x)}{}^{(y)}\phi_{(y)}{}^{(z)}=\psi_{(x)}{}^{(z)}.\end{aligned}$$

Applying the operator $\bar{\nabla}_\gamma = B_\gamma{}^\kappa D_\kappa$ to (4.30), (4.31) and (4.32) and using (4.5), (4.13)$\sim$(4.15) and (4.17), we also have

$$(4.34)\qquad \begin{aligned}&\bar{\nabla}_\gamma \phi_\beta{}^\alpha = \xi_\beta \delta_\gamma{}^\alpha - \xi^\alpha g_{\gamma\beta} + p_\beta A_\gamma{}^\alpha - p^\alpha A_{\gamma\beta},\\ &\bar{\nabla}_\gamma \psi_\beta{}^\alpha = \eta_\beta \delta_\gamma{}^\alpha - \eta^\alpha g_{\gamma\beta} + q_\beta A_\gamma{}^\alpha - q^\alpha A_{\gamma\beta},\\ &\bar{\nabla}_\gamma \theta_\beta{}^\alpha = \zeta_\beta \delta_\gamma{}^\alpha - \zeta^\alpha g_{\gamma\beta} + r_\beta A_\gamma{}^\alpha - r^\alpha A_{\gamma\beta},\end{aligned}$$

$$(4.35)\qquad \begin{aligned}&\bar{\nabla}_\beta p_\alpha = -A_{\beta\gamma}\phi_\alpha{}^\gamma,\ \ \bar{\nabla}_\beta p^\alpha = A_\beta{}^\gamma \phi_\gamma{}^\alpha,\ \ \bar{\nabla}_\beta q_\alpha = -A_{\beta\gamma}\psi_\alpha{}^\gamma,\\ &\bar{\nabla}_\beta q^\alpha = A_\beta{}^\gamma \psi_\gamma{}^\alpha,\ \ \bar{\nabla}_\beta r_\alpha = -A_{\beta\gamma}\theta_\alpha{}^\gamma,\ \ \bar{\nabla}_\beta r^\alpha = A_\beta{}^\gamma \theta_\gamma{}^\alpha,\end{aligned}$$

$$(4.36)\qquad \begin{aligned}&A_{\beta\alpha}{}^{(y)}\phi_{(y)}{}^{(x)} = A_{\beta\gamma}{}^{(x)}\phi_\alpha{}^\gamma + l_\beta{}^{(x)} p_\alpha,\\ &A_{\beta\alpha}{}^{(y)}\psi_{(y)}{}^{(x)} = A_{\beta\gamma}{}^{(x)}\psi_\alpha{}^\gamma + l_\beta{}^{(x)} q_\alpha,\\ &A_{\beta\alpha}{}^{(y)}\theta_{(y)}{}^{(x)} = A_{\beta\gamma}{}^{(x)}\theta_\alpha{}^\gamma + l_\beta{}^{(x)} r_\alpha,\end{aligned}$$

$$(4.37)\qquad \begin{aligned}&A_{\beta\gamma}{}^{(x)}p^\gamma = -l_\beta{}^{(y)}\phi_{(y)}{}^{(x)},\ \ A_{\beta\gamma}{}^{(x)}q^\gamma = -l_\beta{}^{(y)}\psi_{(y)}{}^{(x)},\\ &A_{\beta\gamma}{}^{(x)}r^\gamma = -l_\beta{}^{(y)}\theta_{(y)}{}^{(x)},\end{aligned}$$

$$(4.38)\qquad \begin{aligned}&\bar{\nabla}_\beta \phi_{(y)}{}^{(x)} = l_{\beta(y)}{}^{(z)}\phi_{(z)}{}^{(x)} - l_{\beta(z)}{}^{(x)}\phi_{(y)}{}^{(z)},\\ &\bar{\nabla}_\beta \psi_{(y)}{}^{(x)} = l_{\beta(y)}{}^{(z)}\psi_{(z)}{}^{(x)} - l_{\beta(z)}{}^{(x)}\psi_{(y)}{}^{(z)},\\ &\bar{\nabla}_\beta \theta_{(y)}{}^{(x)} = l_{\beta(y)}{}^{(z)}\theta_{(z)}{}^{(x)} - l_{\beta(z)}{}^{(x)}\theta_{(y)}{}^{(z)}\end{aligned}$$

Differentiating (4.5) covariantly along $\bar{M}$ and using (4.13) and (4.30)$\sim$(4.32), we find

$$(4.39)\qquad \bar{\nabla}_\beta \xi^\alpha = \phi_\beta{}^\alpha,\ \ \bar{\nabla}_\beta \eta^\alpha = \psi_\beta{}^\alpha,\ \ \bar{\nabla}_\beta \zeta^\alpha = \theta_\beta{}^\alpha,$$

$$(4.40)\qquad A_{\beta\alpha}\xi^\alpha = p_\beta,\ \ A_{\beta\alpha}\eta^\alpha = q_\beta,\ \ A_{\beta\alpha}\zeta^\alpha = r_\beta,$$

$$(4.41)\qquad A_{\beta\alpha}{}^{(x)}\xi^\alpha = 0,\ \ A_{\beta\alpha}{}^{(x)}\eta^\alpha = 0,\ \ A_{\beta\alpha}{}^{(x)}\zeta^\alpha = 0.$$

On the other hand, by differentiating (4.6) covariantly and using (4.8), (4.9), (4.10), (4.12), (4.29), (4.33) and (4.39), we have

$$(4.42)\qquad \begin{aligned}&\phi_j{}^h = -h_j{}^h{}_s a^s,\ \ \psi_j{}^h = -h_j{}^h{}_s b^s,\ \ \theta_j{}^h = -h_j{}^h{}_s c^s,\\ &h_j{}^h{}_s = -(\phi_j{}^h a_s + \psi_j{}^h b_s + \theta_j{}^h c_s).\end{aligned}$$

If we apply the operator $\nabla_j = B_j{}^A \tilde{\nabla}_A = E^\alpha{}_j \bar{\nabla}_\alpha = B_j{}^B E^\kappa{}_B D_\kappa$ to (4.4) and use (2.12), (4.11)$\sim$(4.13), (4.20) and (4.42), then we have

$$\begin{aligned}&A_{jh}E_\alpha{}^h C^A + A_{jh}{}^{(x)}E_\alpha{}^h C_{(x)}{}^A + h_j{}^h{}_s C_\alpha{}^s B_h{}^A\\ &= h_B{}^A{}_s C_\kappa{}^s B_\alpha{}^\kappa B_j{}^B + A_{\beta\alpha}E^\beta{}_j C^A + A_{\beta\alpha}{}^{(x)}E^\beta{}_j C_{(x)}{}^A,\end{aligned}$$

and consequently

$$(4.43)\qquad A_{jk} = A_{\beta\alpha}E^\beta{}_j E^\alpha{}_k,\ \ A_{jk}{}^{(x)} = A_{\beta\alpha}{}^{(x)}E^\beta{}_j E^\alpha{}_k,$$

$$(4.44)\qquad h_j{}^h{}_s C_\alpha{}^s B_h{}^A = h_B{}^A{}_s C_\kappa{}^s B_\alpha{}^\kappa B_j{}^B.$$

Transvecting $E_\gamma{}^j E_\delta{}^k$ to (4.43) and using (4.9), (4.33), (4.40), (4.41),

$a_sC_\alpha{}^s=\xi_\alpha$, $b_sC_\alpha{}^s=\eta_\alpha$ and $c_sC_\alpha{}^s=\zeta_\alpha$ and replacing the indices γ and δ with β and α respectively, we get

$$(4.45)\quad A_{\beta\alpha}=A_{jk}E_\beta{}^jE_\alpha{}^k+(p_\beta\xi_\alpha+q_\beta\eta_\alpha+r_\beta\zeta_\alpha)+(p_\alpha\xi_\beta+q_\alpha\eta_\beta+r_\alpha\zeta_\beta),$$
$$(4.46)\quad A_{\beta\alpha}{}^{(x)}=A_{jk}{}^{(x)}E_\beta{}^jE_\alpha{}^k.$$

Thus we have the following lemma from (2.9)~(2.11), (3.8), (4.37) and (4.46).

LEMMA 4.1. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with distinguished normal C^A of a quaternionic projective space QP^m admitting an almost contact metric compound 3-structure. If the second fundamental tensors of M are commutative with the structure tensors ϕ, ψ and θ respectively induced on M, then we have*

$$(4.47)\quad A_{\beta\alpha}{}^{(x)}=0,\quad l_{\beta(x)}=0.$$

Since the ambient manifold S^{4m+3} for $\overline{M}$ is a space of constant curvature 1, the equation of Gauss for $\overline{M}$ are given by

$$(4.48)\quad K_{\delta\gamma\beta}{}^\alpha=\delta_\delta{}^\alpha g_{\gamma\beta}-\delta_\gamma{}^\alpha g_{\delta\beta}+A_\delta{}^\alpha A_{\gamma\beta}-A_\gamma{}^\alpha A_{\delta\beta}+A_\delta{}^{\alpha(x)}A_{\gamma\beta(x)}+A_\gamma{}^{\alpha(x)}A_{\delta\beta(x)},$$

where $K_{\delta\gamma\beta}{}^\alpha$ is the Riemann-Christoffel curvature tensor of $\overline{M}$, those of Codazzi by

$$(4.49)\quad \overline{\nabla}_\gamma A_{\beta\alpha}-\overline{\nabla}_\beta A_{\gamma\alpha}-l_{\gamma(x)}A_{\beta\alpha}{}^{(x)}+l_{\beta(x)}A_{\gamma\alpha}{}^{(x)}=0,$$
$$(4.50)\quad \overline{\nabla}_\gamma A_{\beta\alpha}{}^{(x)}-\overline{\nabla}_\beta A_{\gamma\alpha}{}^{(x)}+l_\gamma{}^{(x)}A_{\beta\alpha}-l_\beta{}^{(x)}A_{\gamma\alpha}+l_{\gamma(y)}{}^{(x)}A_{\beta\alpha}{}^{(y)}-l_{\beta(y)}{}^{(x)}A_{\gamma\alpha}{}^{(y)}=0,$$

and those of Ricci by

$$(4.51)\quad \overline{\nabla}_\beta l_\alpha{}^{(x)}-\overline{\nabla}_\alpha l_\beta{}^{(x)}+A_\beta{}^\gamma A_{\gamma\alpha}{}^{(x)}-A_\alpha{}^\gamma A_{\beta\gamma}{}^{(x)}+l_{\beta(y)}{}^{(x)}l_\alpha{}^{(y)}-l_{\alpha(y)}{}^{(x)}l_\beta{}^{(y)}=0,$$
$$(4.52)\quad \overline{\nabla}_\beta l_{\alpha(x)}{}^{(y)}-\overline{\nabla}_\alpha l_{\beta(x)}{}^{(y)}+A_\beta{}^\gamma{}_{(x)}A_{\alpha\gamma}{}^{(y)}-A_\alpha{}^\gamma{}_{(x)}A_{\beta\gamma}{}^{(y)}+l_{\beta(x)}l_\alpha{}^{(y)}-l_{\alpha(x)}l_\beta{}^{(y)}+l_{\beta(z)}{}^{(y)}l_{\alpha(x)}{}^{(z)}-l_{\alpha(z)}{}^{(y)}l_{\beta(x)}{}^{(z)}=0.$$

We now assume that the second fundamental tensors of a base space M for $\overline{M}$ commute with the structure tensors $\phi_j{}^h$, $\psi_j{}^h$ and $\theta_j{}^h$ of the submersion π, that is, (3.1), (3.2) and (3.3) hold. Then we can easily verify that the second fundamental tensors of the total space $\overline{M}$ also commute with $\phi_\beta{}^\alpha$, $\psi_\beta{}^\alpha$ and $\theta_\beta{}^\alpha$ because of (4.29), (4.33), (4.45) and

(4. 46), that is,

$$(4.53) \qquad A_{\beta\gamma}\phi_\alpha{}^\gamma + A_{\alpha\gamma}\phi_\beta{}^\gamma = 0, \quad A_{\beta\gamma}{}^{(x)}\phi_\alpha{}^\gamma + A_{\alpha\gamma}{}^{(x)}\phi_\beta{}^\gamma = 0,$$
$$(4.54) \qquad A_{\beta\gamma}\psi_\alpha{}^\gamma + A_{\alpha\gamma}\psi_\beta{}^\gamma = 0, \quad A_{\beta\gamma}{}^{(x)}\psi_\alpha{}^\gamma + A_{\alpha\gamma}{}^{(x)}\psi_\beta{}^\gamma = 0,$$
$$(4.55) \qquad A_{\beta\gamma}\theta_\alpha{}^\gamma + A_{\alpha\gamma}\theta_\beta{}^\gamma = 0, \quad B_{\beta\gamma}{}^{(x)}\theta_\alpha{}^\gamma + A_{\alpha\gamma}{}^{(x)}\psi_\beta{}^\gamma = 0.$$

Transvecting (4. 45) with p^α, q^α and r^α and using (3. 4), (4. 29) and (4. 33), we have

$$(4.56) \qquad A_{\beta\alpha}p^\alpha = \alpha p_\beta + \xi_\beta, \quad A_{\beta\alpha}q^\alpha = \alpha q_\beta + \eta_\beta, \quad A_{\beta\alpha}r^\alpha = \alpha r_\beta + \zeta_\beta$$

respectively.

LEMMA 4. 2. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a quaternionic projective space QP^m admitting an almost contact metric compound 3-structure. If the second fundamental tensors of M are commutative with the structure tensors of the submersion π, then we have*

$$(4.57) \qquad A_{\beta\gamma}A_\alpha{}^\gamma = \alpha A_{\beta\alpha} + g_{\beta\alpha}.$$

Proof. Transvecting (4. 46) with $A_\gamma{}^\alpha = A_{hi}E_\gamma{}^h E^{\alpha i} + (p_\gamma\xi^\alpha + q_\gamma\eta^\alpha + r_\gamma\zeta^\alpha) + (p^\alpha\xi_\gamma + q^\alpha\eta_\gamma + r^\alpha\zeta_\gamma)$ and taking account of (3. 4), (4. 8), (4. 29) and (4. 33), we obtain (4. 57) with the aid of (4. 8), (4. 13), (4. 29) and (4. 45).

LEMMA 4. 3. *Under the same assumptions as those stated in Lemma* 4. 2, *we obtain*

$$(4.58) \qquad \bar{\nabla}_\gamma A_{\beta\alpha} = 0.$$

Proof. Applying the operator $\nabla_k = E^\gamma{}_k\bar{\nabla}_\gamma$ to (4. 45), we have

$$\begin{aligned} E^\gamma{}_k\bar{\nabla}_\gamma A_{\beta\alpha} = & (\nabla_k A_{ji} + p_i\phi_{kj} + q_i\psi_{kj} + r_i\theta_{kj} + p_j\phi_{ki} + q_j\psi_{ki} + r_j\theta_{ki})E_\beta{}^j E_\alpha{}^i \\ & - (A_{kh}\phi_i{}^h + A_{ih}\phi_k{}^h)(E_\beta{}^i\xi_\alpha + E_\alpha{}^i\xi_\beta) \\ & - (A_{kh}\psi_i{}^h + A_{ih}\psi_k{}^h)(E_\beta{}^i\eta_\alpha + E_\alpha{}^i\eta_\beta) \\ & - (A_{kh}\theta_i{}^h + A_{ih}\theta_k{}^h)(E_\beta{}^i\zeta_\alpha + E_\alpha{}^i\zeta_\beta), \end{aligned}$$

because of (4. 12), (4. 29), (4. 35), (4. 39), (4. 42) and (4. 45), from which, using (3. 1) ~ (3. 3) and (3. 14), we have

$$(4.59) \qquad E^\gamma{}_k\bar{\nabla}_\gamma A_{\beta\alpha} = 0.$$

On the other hand, by (4.47), we can have from (4.49)

$$(4.60)\qquad \bar{\nabla}_\gamma A_{\beta\alpha}-\bar{\nabla}_\beta A_{\gamma\alpha}=0.$$

Transvecting (4.59) with $E_\delta{}^k$, we have

$$(4.61)\qquad \bar{\nabla}_\delta A_{\beta\alpha}=\bar{\nabla}_\gamma A_{\beta\alpha}(\xi^\gamma\xi_\delta+\eta^\gamma\eta_\delta+\zeta^\gamma\zeta_\delta)$$

Differentiating (4.40) and making use of (4.35), (4.39), (4.53), (4.54), (4.55) and (4.60), we have

$$\xi^\gamma(\bar{\nabla}_\gamma A_{\beta\alpha})=0,\quad \eta^\gamma(\bar{\nabla}_\gamma A_{\beta\alpha})=0,\quad \zeta^\gamma(\bar{\nabla}_\gamma A_{\beta\alpha})=0,$$

from which, using (4.61), $\bar{\nabla}_\delta A_{\beta\alpha}=0$. Therefore, Lemma 4.3 is proved.

We consider the identity:

$$\frac{1}{2}\Delta(A_{\beta\alpha}{}^x A^{\beta\alpha}{}_x)=(\bar{\nabla}^\gamma\bar{\nabla}_\gamma A_{\beta\alpha}{}^x)A^{\beta\alpha}{}_x+\|\bar{\nabla}_\gamma A_{\beta\alpha}{}^x\|^2,$$

where $\Delta=g^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta$ and $A_{\beta\alpha}{}^{1*}=A_{\beta\alpha 1*}=A_{\beta\alpha}$. From this identity we can see that the second fundamental tensors $A_{\beta\alpha}{}^x$ are parallel because of (4.47) and (4.58). Thus the first normal space $N_1(\bar{p})$ defined to be the orthogonal complement of $\{C_x{}^\kappa\in T_{\bar{p}}{}^\perp(\bar{M})\mid A_{C_x}\kappa=0\}$ in $T_{\bar{p}}{}^\perp(\bar{M})$ is invariant under parallel translation with respect to connection in the normal bundle and of constant dimension 1, where $A_{C_x}\kappa$ are second fundamental tensors associated with $C_x{}^\varepsilon$ and $T_{\bar{p}}{}^\perp(\bar{M})$ is the normal space at $\bar{p}\in\bar{M}$. Thus, by the reduction theorem, we conclude the total space $\bar{M}$ for M is contained in an $(n+4)$-dimensional unit sphere $S^{n+4}(\subset S^{4m+3})$ and consequently the base space M is contained as a hypersurface in a quaternionic projective space $QP^{(n+1)/4}$ of real dimension $n+1$. And hence the diagram in the beginning in §4 reduces to

$$\begin{array}{ccc} \bar{M}^{n+3} & \xrightarrow{\ \tilde{i}\ } & S^{n+4}\subset S^{4m+3} \\ \pi\downarrow & & \downarrow\tilde{\pi} \\ M^n & \xrightarrow[\ i\ ]{} & QP^{(n+1)/4}\subset QP^m \end{array}$$

Thus we have

THEOREM 4.4. *Let M be an $n(>1)$-dimensional semi-invariant submanifold with the distinguished normal C^A of a quaternionic projective space QP^m admitting an almost contact metric compound 3-structure. If the* *second fundamental tensor are commutative with the structure tensors of the submersion π, then M is contained as a hypersuface in a quaternionic projective space $QP^{(n+1)/4}$ of real dimension $n+1$.*

Denoting by x the eigenvalue corresponding to an eigenvector of $A_\beta{}^\alpha$, (4.57) implies $x^2-ax-1=0$. Then we see thet $A_\beta{}^\alpha$ has exactly two constant eigenvalues $x_1=(\alpha+\sqrt{\alpha^2+4})/2$ and $x_2=(\alpha-\sqrt{\alpha^2+4})/2$. On the other hand, transvecting (4.57) with ξ^α, η^α and ζ^α and using (4.40), we have respectively

$$A_{\beta\gamma}p^\gamma=\alpha p_\beta+\xi_\beta,\quad A_{\beta\gamma}q^\gamma=\alpha q_\beta+\eta_\beta,\quad A_{\beta\gamma}r^\gamma=\alpha r_\beta+\zeta_\beta,$$

from which, taking account of $x_1{}^2=\alpha x_1+1$,

$$A_\beta{}^\alpha(x_1p^\beta+\xi^\beta)=x_1(x_1p^\alpha+\xi^\alpha),\quad A_\beta{}^\alpha(x_1q^\beta+\eta^\beta)=x_1(x_1q^\alpha+\eta^\alpha),$$
$$A_\beta{}^\alpha(x_1r^\beta+\zeta^\beta)=x_1(x_1r^\alpha+\zeta^\alpha).$$

Therefore $x_1p^\alpha+\xi^\alpha$, $x_1q^\alpha+\eta^\alpha$ and $x_1r^\alpha+\zeta^\alpha$, which will be denoted by $e_1{}^\alpha, e_2{}^\alpha$ and $e_3{}^\alpha$ respectively, are eigenvectors of $A_\beta{}^\alpha$ corresponding to x_1, where $e_1{}^\alpha, e_2{}^\alpha$ and $e_3{}^\alpha$ are mutually orthogonal because of (4.33). Assuming that there exists another eigenvector $e_4{}^\alpha$ of $A_\beta{}^\alpha$ corresponding to x_1 and supposing that $e_4{}^\alpha$ is orthogonal to $e_1{}^\alpha, e_2{}^\alpha$ and $e_3{}^\alpha$, we have

$$x_1(p_\alpha e_4{}^\alpha)+(\xi_\alpha e_4{}^\alpha)=0,\quad x_1(q_\alpha e_4{}^\alpha)+(\eta_\alpha e_4{}^\alpha)=0,$$
$$x_1(r_\alpha e_4{}^\alpha)+(\zeta_\alpha e_4{}^\alpha)=0. \tag{4.62}$$

From (4.40) and $A_\beta{}^\alpha e_4{}^\beta=x_1e_4{}^\alpha$, we have

$$(p_\alpha e_4{}^\alpha)-x_1(\xi_\alpha e_4{}^\alpha)=0,\quad (q_\alpha e_4{}^\alpha)-x_1(\eta_\alpha e_4{}^\alpha)=0,$$
$$(r_\alpha e_4{}^\alpha)-x_1(\zeta_\alpha e_4{}^\alpha)=0. \tag{4.63}$$

The last two equations yield

$$p_\alpha e_4{}^\alpha=q_\alpha e_4{}^\alpha=r_\alpha e_4{}^\alpha=0,\quad \xi_\alpha e_4{}^\alpha=\eta_\alpha e_4{}^\alpha=\zeta_\alpha e_4{}^\alpha=0 \tag{4.64}$$

because of $x_1{}^2+1\neq0$.

From the first relationships of (4.53), (4.54) and (4.55), we find

$$A_\gamma{}^\beta(\phi_\alpha{}^\gamma e_4{}^\alpha)=x_1(\phi_\alpha{}^\beta e_4{}^\alpha),\quad A_\gamma{}^\beta(\psi_\alpha{}^\gamma e_4{}^\alpha)=x_1(\psi_\alpha{}^\beta e_4{}^\alpha),$$
$$A_\gamma{}^\beta(\theta_\alpha{}^\gamma e_4{}^\alpha)=x_1(\theta_\alpha{}^\beta e_4{}^\alpha).$$

Thus, $\phi_\alpha{}^\beta e_4{}^\alpha$, $\psi_\alpha{}^\beta e_4{}^\alpha$ and $\theta_\alpha{}^\beta e_4{}^\alpha$ are also eigenvector of $A_\alpha{}^\beta$ corresponding to x_1, which are mutually orthogonal and also orthogonal to $e_1{}^\alpha$, $e_2{}^\alpha$, $e_3{}^\alpha$

and $e_4{}^{\alpha}$ because of (4.33) and (4.64). Hence multiplicity of the eigenvalue x_1 is necessarily $4p+3$ for some integer p. Similarly we can prove that multiplicity of x_2 is $4q+3$, where $q=(n-3)/4-p$. From this fact and (4.58), the eigenspaces corresponding to x_1 and x_2 define respecively $(4p+3)$ and $(4q+3)$-dimensional distributions Dx_1 and Dx_2 over $\overline{M}$ and those distributions are both integrable and parallel. Moreover each integral manifold of Dx_1 and Dx_2 is totally godesic in $\overline{M}$ and totally umbilical in Q^{m+1}. Making use of a usual manner (cf. [11]), we obtain

$$\overline{M}=S^{4p+3}(a)\times S^{4q+3}(b),$$

(p, q) being some portion of $(n-3)/4$ and $a^2+b^2=1$.

Thus we have the following

THEOREM 4.5. *Let M be an $n(>1)$-dimensional complete semi-invariant submanifold with the distinguished normal C^A of a quaternionic projective space QP^m admitting an almost contact metric compound 3-structure. If the second fundamental tensors are commutative with the structure tensors of the submersion π, then M is the model space $M_{p,q}^{Q}$ (a, b), where (p, q) is some portion of $(n-3)/4$ and $a^2+b^2=1$.*

COROLLARY 4.6. ([13]) *Let M be a complete real hypersurface of a quaternionic projective space QP^m. If the second fundamental tensors are commutative with the structure tensors of the submersion π, then M is the model space $M_{p,q}^{Q}$ (a, b).*

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