Comm. Korean Math. Soc. 2 (1987), No. 1, pp. 71~87

NONLINEAR ERGODIC THEOREMS FOR NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

Kwang Pak Park and Jong Kyu Kim

I. Introduction

In this paper, we are going to extend the reresult of N. Hirano ([10]) for a nonexpansive semigroup in a uniformly convex Banach space which has a Fréchet differentiable norm. That is to say, we will prove the existence of the weak limit of the Cesàro means

$$A_tS(h)x=\frac{1}{t}\int_0^t S(s+h)x\,ds$$

uniformly in $h\geqq 0$. Analogous problems were studied in [2], [13] and [20].

Some rudiments in the geometry of Banach spaces are necessary for the proof of the main theorem of this paper.

Let X be a Banach space and X^* its dual. The value of $x^*\in X^*$ at $x\in X$ will be denoted by $\langle x, x^*\rangle$. With each x in X, we associate the set

$$J(x)=\{x^*\in X^*: \langle x, x^*\rangle=\|x\|^2=\|x^*\|^2\}.$$

Using the Hahn-Banach theorem it is immediatedly clear that $J(x)\neq\phi$ for any x in X. Then the multi-valued operator $J: X\to X^*$ is called the duality mapping of X. Let $B=\{x\in X: \|x\|=1\}$ stand for the unit sphere of X. Then, the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each x and y in B. It is said to be Fréchet differentiable if for each x in B, this limit is attained uniformly for y in B. It is well known that if X is smooth, then the duality mapping J is single-valued. And we also know that if the norm of X is Fréchet differentiable, then J is norm to norm continuous ([1], [6] and [7]). Let C be a closed convex subset of X. A family $\{S(t) : t\geqq 0\}$ of mappings from C into itself is called a nonexpansive semigroup on C if

(I) $S(t+s)=S(t)S(s)$ for all $t, s\geqq 0$,

(II) $S(0)=I$ (identity),

(III) $\lim_{t\to 0} S(t)x=x$ for every $x\in C$,

(IV) $\|S(t)x-S(t)y\|\leqq\|x-y\|$ for all x, y in C and $t\geqq 0$.

For a subset D of X, conv$D$ denotes the convex hull of D, and $\bar{D}$ the closure of D. Let $F(S)=\bigcap_{t\geqq 0}F(S(t))$ be the set of all common fixed points of $\{S(t) : t\geqq 0\}$.

For x and y in X, sgn$[x, y]$ denotes the set

$$\{\lambda x+(1-\lambda)y : 0\leqq\lambda\leqq 1\}.$$

In this paper, unless otherwise specfied, X will denote a uniformly convex Banach space with modulus of convexity δ. The modulus of convexity of X is the function $\delta : [0,2]\to[0,1]$ defined by

$$\delta(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2} : \|x\|\leqq 1,\ \|y\|\leqq 1,\ \|x-y\|\geqq\varepsilon\right\}$$

for $0\leqq\varepsilon\leqq 2$. X is uniformly convex if and only if $\delta(\varepsilon)>0$ for $\varepsilon>0$ ([3], [6], [17] and [19]). It is shown in [9], [18] and [23] that δ is nondecreasing. Hence, if X is uniformly convex and $\delta(\varepsilon_n)\to 0$, then $\varepsilon_n\to 0$. Furthermore

$$\delta(\varepsilon)\leqq 1-\sqrt{1-\frac{\varepsilon^2}{4}}\ ([6]).$$

It is well known that if X is a uniformly convex Banach space, then the set $F(S)$ is nonempty ([1]) and a closed convex subset of C ([6]).

We define the Cesàro mean $A_t x$ by

$$A_t x=\frac{1}{t}\int_0^t S(s)x\,ds \text{ (for the continuous case)}$$

for all $x\in C$, $t>0$.

In this paper, we establish a convergence theorem for a nonexpansive semigroup in the framework of a uniformly convex Banach space. In Banach spaces, Hilbert space techniques as seen in G. Rodé ([21]) can not play a role. Therefore, we shall introduce compact means, which should be compared with measures with compact, support, to obtain the result.

We denote by $B([0,\infty))$ [resp. $CB([0,\infty))$] the Banach space of all bounded [resp. bounded continous] real valued functions on $[0,\infty)$ with supremum norm.

For $s\geqq 0$ and $f\in B([0,\infty))$ [resp. $CB([0,\infty))$], we define an element $r_s f$ in $B([0,\infty))$ [resp. $CB([0,\infty))$] given by $r_s f(t)=f(t+s)$ for all $t\geqq 0$. The mapping $r_s: f\to r_s f$ is a continuous linear operator in $B([0,\infty))$ [resp. $CB([0,\infty))$] for all $s\geqq 0$. An element $\mu\in B([0,\infty))^*$ is called a mean on $[0,\infty)$ if $\|\mu\|=\mu(1)=1$ ([12], [14], [15] and [22]). For every $f\in B([0,\infty))$ and $\mu\in B([0,\infty))^*$, we denote the value of μ at f by $\mu(f)$ or

$$\int_0^\infty f(s)\,d\mu(s)$$

to specify the variable s of f. A mean μ on $[0,\infty)$ is said to be compact if there exists a compact subset K of $[0,\infty)$ such that $\mu(1_k)=1$, where 1_k is a real valued function on $[0,\infty)$ with value 1 on K and 0 elsewhere. Especially, a compact mean μ is said to be finite if the compact subset K consists of finite points. If μ is a finite mean on $[0,\infty)$, then it follows that μ is expressed by $\sum_{i=1}^n a_i\delta_{s_i}$ for some $s_i\geqq 0$ and $a_i\geqq 0$ $(i=1,2,3,\cdots,n)$ such that $\sum_{i=1}^n a_i=1$, where δ_t is a mean on $[0,\infty)$ defined by $\delta_t(f)=f(t)$ for all $f\in B([0,\infty))$. A mean μ on $[0,\infty)$ is said to be invariant [resp. c-invariant] if $\mu(r_s f)=\mu(f)$ for all $f\in B([0,\infty))$ [resp. $CB([0,\infty))$] and $s\geqq 0$. Therefore this definition agrees with the M.M. Day's definition of finite means in [4].

Suppose that the set $\{S(t)x: t\geqq 0\}$ is bounded for all $x\in C$. Then, for a mean μ on $[0,\infty)$ and $x\in C$, we can define a continuous functional ϕ_x on X^* by

$$\phi_x(x^*)=\int_0^\infty\langle S(t)x, x^*\rangle\,d\mu(t)$$

for each $x^*\in X^*$. Since C is a closed convex subset and X is reflexive, by the Hahn-Banach theorem, ϕ_x is expressed by an element in C, which is denoted by $\mathcal{T}_\mu x$ or

$$\int_0^\infty S(t)x d\mu(t)$$

to specify the variable t. If μ is finite, say

$$\mu=\sum_{i=1}^n a_i\delta_{s_i} \quad (s_i \geqq 0, \ \ a_i \geqq 0, \ \ i=1,2,3,\cdots,n, \ \ \sum_{i=1}^n a_i=1)$$

then

$$\mathcal{T}_\mu x=\sum_{i=1}^n a_i S(s_i)x.$$

If μ is a compact mean on $[0,\infty)$, $X\in C$ and $\varepsilon>0$, then there exists a finite mean λ on $[0,\infty)$ such that

$$\|\mathcal{T}_\mu S(t)x-\mathcal{T}_\lambda S(t)x\|<\varepsilon$$

for all $t\geqq 0$ ([11]).

II. Lemmas and Proposition

The following lemmas and proposition are crucial for our results. The next Lemma 2.1 is known ([8]). It is simple consequence of the condition of the modulus of convexity.

LEMMA 2.1. *Let x and y be in X. If $\|x\|\leqq r$, $\|y\|\leqq r$, $r\leqq R$ and $\|x-y\|\geqq\varepsilon>0$, then*

$$\|\lambda x+(1-\lambda)y\|\leqq r(1-2\lambda(1-\lambda)\delta_R(\varepsilon))$$

for all $0\leqq\lambda\leqq 1$, where $\delta_R(\varepsilon)=\delta\left(\frac{\varepsilon}{R}\right)$.

The proofs of our following lemmas are based on methods used in [16].

LEMMA 2.2. *Let C be a closed convex subset of X and $\{S(t):t\geqq 0\}$ a nonexpansive semigroup on C. Let x be in C, $f\in F(S)$ and $0<\alpha\leqq\beta<1$. Then for each $\varepsilon>0$, there exists $t_0\geqq 0$ such that for all $t\geqq t_0$,*

$$\|S(h)(\lambda S(t)x+(1-\lambda)f)-(\lambda S(t+h)x+(1-\lambda)f)\|\leqq\varepsilon$$

for all $h>0$ *and* $\alpha\leqq\lambda\leqq\beta$.

Proof. Let $r=\lim_{t\to\infty}\|S(t)x-f\|$, $R=\|x-f\|$, and $c=\min\{2\lambda(1-\lambda):\alpha\leqq\lambda\leqq\beta\}$. Since δ is nondecreasing, for given $\varepsilon>0$, we can choose $d>0$ so small that

$$\frac{r}{(r+d)}>1-c\delta\left(\frac{\varepsilon}{r+d}\right),$$

where δ is the modulus of convexity of the norm. And also, there exists $t_0\geqq 0$ such that for all $t\geqq t_0$,

$$\|S(t)x-f\|\leqq r+d.$$

For $t\geqq t_0$, $h>0$ and $\alpha\leqq\lambda\leqq\beta$, we put $u=(1-\lambda)(S(h)z-f)$ and $v=\lambda(S(t+h)x-S(h)z)$ where $z=\lambda S(t)x+(1-\lambda)f$. Then we have

$$\|S(h)(\lambda S(t)x+(1-\lambda)f)-(\lambda S(t+h)x+(1-\lambda)f)\|=\|u-v\|,$$
$$\|u\|\leqq(1-\lambda)\|z-f\|=\lambda(1-\lambda)\|S(t)x-f\|\leqq\lambda(1-\lambda)(r+d)$$

and

$$\|v\|\leqq\lambda\|S(t)x-z\|\leqq\lambda(1-\lambda)(r+d).$$

Suppose that $\|u-v\|\geqq\varepsilon$, for some $\varepsilon\geqq 0$, then by Lemma 2. 1, we have

$$\begin{aligned}\lambda(1-\lambda)\|S(t+h)x-f\|&=\|\lambda u+(1-\lambda)v\|\\&\leqq\lambda(1-\lambda)(r+d)\left(1-2\lambda(1-\lambda)\delta\left(\frac{\varepsilon}{r+d}\right)\right)\\&\leqq\lambda(1-\lambda)(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right).\end{aligned}$$

Hence we have,

$$(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right)<r\leqq(r+d)\left(1-c\delta\left(\frac{\varepsilon}{r+d}\right)\right)$$

which is a contradiction. Therefore, for all $t\geqq t_0$

$$\|S(h)(\lambda S(t)x+(1-\lambda)f)-(\lambda S(s+h)x+(1-\lambda)f)\|<\varepsilon$$

for all $h>0$ and $\alpha\leqq\lambda\leqq\beta$.

LEMMA 2.3. *Let C be a closed convex subset of X and $\{S(t)x : t\geq 0\}$ a bounded set in C. Let $z\in W(x)=\bigcap_{t_0\geqq 0}\overline{conv}\{S(t)x : t\geqq t_0\}$, $y\in C$ and $\{p_t\}$ a net of elements in C with $p_t\in \mathrm{sgn}[y, S(t)x]$ and $\|p_t-z\|=\min\{\|u-z\| : u\in \mathrm{sgn}[y, S(t)x]\}$. If $\{p_t\}$ couverges strongly to y as $t\to\infty$, then $y=z$.*

Proof. Since the duality mapping J is single valued, it follows from Theorem 2.5 in [5]

$$\langle u-p_t,\ J(p_t-z)\rangle \geqq 0$$

for all $u\in \mathrm{sgn}\ [y, S(t)x]$. Putting $u=S(t)x$, we have

$$\langle S(t)x-p_t, J(p_t-z)\rangle \geqq 0.$$

Since, $\lim_{t\to\infty} p_t=y$ and $\{S(t)x : t\geq 0\}$ is bounded, there exists a $K>0$ and $t_0\geqq 0$ such that

$$\|S(t)x-y\|\leqq K \text{ and } \|p_t-z\|\leqq K$$

for all $t\geqq t_0$. Let $\varepsilon>0$ and choose $\delta>0$ and choose $\delta>0$ so small that $2\delta K<\varepsilon$. Since the norm of X is Fréchet differentiable, J is norm to norm continuous, and so we can choose $t'\geqq t_0$ such that for all $t\geqq t'$, $\|p_t-y\|\leqq\delta$ and $\|J(p_t-z)-J(y-z)\|\leqq\delta$. Since for $t\geqq t'$,

$$\begin{aligned}
&|\langle S(t)x-p_t,\ J(p_t-z)\rangle-\langle S(t)x-y,\ J(y-z)\rangle|\\
&\quad=|\langle S(t)x-p_t,\ J(p_t-z)\rangle-\langle S(t)x-y,\ J(p_t-z)\rangle\\
&\qquad+\langle S(t)x-y,\ J(p_t-z)\rangle-\langle S(t)x-y,\ J(y-z)\rangle|\\
&\quad\leqq\|p_t-y\|\|p_t-z\|+\|S(t)x-y\|\|J(p_t-z)-J(y-z)\|\\
&\quad\leqq\delta K+\delta K=2\delta K<\varepsilon,
\end{aligned}$$

we have

$$\begin{aligned}
\langle S(t)x-y,\ J(y-z)\rangle &\geqq\langle S(t)x-p_t,\ J(p_t-z)\rangle-\varepsilon\\
&\geqq 0-\varepsilon=-\varepsilon.
\end{aligned}$$

Since $z\in W(x)$, we have $\langle z-y,\ J(y-z)\rangle\geqq-\varepsilon$. This implies $\|z-y\|=0$ and hence $z=y$.

By using Lemma 2.2 and Lemma 2.3, we can prove the following lemma.

LEMMA 2. 4. *Let C be a closed convex subset of X and $F(S)\neq\phi$. Then for any $z\in\bigcap_{s\geq 0}\overline{conv}\ \{S(t)x : t\geq s\}\cap F(S)$ and $y\in F(S)$, there exists $t_0\geq 0$ such that for all $t\geq t_0$*

$$\langle S(t)x-y,\ J(y-z)\rangle\leq 0.$$

Proof. If $y=z$ or $x=y$, then the result is obvious. So, let $y\neq z$ and $x\neq y$. For any $t\geq 0$, taking a unique element p_t such that $p_t\in\text{sgn}[y, S(t)x]$ and $\|p_t-z\|=\min\ \{\|u-z\| : u\in\text{sgn}\ [y, S(t)x]\}$. Then since $y\neq z$, by Lemma 2. 3, $\{p_t\}$ doesn't converge to y. Hence, we obtain $c>0$ such that for any $t\geq 0$, there is $t_0\geq 0$ with $t_0\geq t$ and $\|p_{t_0}-y\|\geq c$. Setting $p_{t_0}=\alpha_{t_0}S(t_0)x+(1-\alpha_{t_0})y$, $0\leq\alpha_{t_0}\leq 1$, then there exists $c_0>0$ so small that $\alpha_{t_0}\geq c_0$. (in fact, since $x\neq y$, $y\in F(S)$ and $S(t_0)$ is nonexpansive,

$$\begin{aligned} c\leq\|p_{t_0}-y\|&=\|\alpha_{t_0}S(t_0)x+(1-\alpha_{t_0})y-y\|\\ &=\alpha_{t_0}\|S(t_0)x-y\|\leq\alpha_{t_0}\|x-y\|. \end{aligned}$$

Hence, put $c_0=\dfrac{c}{\|x-y\|}$.).

Putting $K=\lim_{t\to\infty}\|S(t)x-y\|$, we have $K>0$. If not, then we have $\lim_{t\to\infty} S(t)x=y$, and so $\lim_{t\to\infty} p_t=y$ which contradicts.

Now, we can choose $\varepsilon>0$ so small that

$$\frac{R}{R+\varepsilon}>1-\delta\left(\frac{c_0K}{R+\varepsilon}\right),$$

where δ is the modulus of convexity of X and $R=\|z-y\|$. Then by Lemma 2. 2, there exists $t'\geq 0$ such that

$$\|S(s)\,(c_0S(t)x+(1-c_0)y)-(c_0S(s+t)x+(1-c_0)y)\|<\varepsilon$$

for all $s\geq 0$. Fix $t_0\geq 0$ with $t_0\geq t'$ and $\|p_{t_0}-y\|\geq c$. Then since $\alpha_{t_0}\geq c_0$, we have

$$\begin{aligned} c_0S(t_0)x+(1-c_0)y&\in\text{sgn}[y, \alpha_{t_0}S(t_0)x+(1-\alpha_{t_0})y]\\ &=\text{sgn}[y, p_{t_0}]. \end{aligned}$$

Letting $c_0S(t_0)x+(1-c_0)y=\lambda y+(1-\lambda)p_{t_0}$ for $0\leq\lambda\leq 1$, then we have

$$\begin{aligned}\|\lambda y+(1-\lambda)p_{t_0}-z\| &= \|\lambda y+(1-\lambda)p_{t_0}-\lambda z-(1-\lambda)z\| \\ &\leqq \lambda\|y-z\|+(1-\lambda)\|p_{t_0}-z\| \\ &\leqq \lambda\|y-z\|+(1-\lambda)\|y-z\| \\ &= \|y-z\|.\end{aligned}$$

Therefore

$$\|c_0S(t_0)x+(1-c_0)y-z\|\leqq\|y-z\|=R.$$

Hence we obtain

$$\begin{aligned}&\|c_0S(s+t_0)x+(1-c_0)y-z\| \\ &\quad\leqq\|S(s)\,(c_0S(t_0)x+(1-c_0)y)-z\|+\varepsilon \\ &\quad\leqq\|c_0S(t_0)x+(1-c_0)y-z\|+\varepsilon \\ &\quad\leqq R+\varepsilon\end{aligned}$$

for all $s\geqq 0$.

On the other hand, since $\|y-z\|=R<R+\varepsilon$ and

$$\begin{aligned}&\|c_0S(s+t_0)x+(1-c_0)y-z)-(y-z)\| \\ &=\|c_0S(s+t_0)x+(1-c_0)y-y\|=c_0\|S(s+t_0)x-y\|\geqq c_0K\end{aligned}$$

for all $s\geqq 0$, by uniform convexity, we have

$$\begin{aligned}&\left\|\frac{1}{2}((c_0S(s+t_0)x+(1-c_0)y-z)+(y-z))\right\| \\ &\leqq(R+\varepsilon)\left(1-\delta\left(\frac{c_0K}{R+\varepsilon}\right)\right)<R\end{aligned}$$

and hence

$$\left\|\frac{c_0}{2}S(s+t_0)x+\left(1-\frac{c_0}{2}\right)y-z\right\|<R$$

for all $s\geqq 0$. Letting $u_s=\frac{c_0}{2}S(s+t_0)x+\left(1-\frac{c_0}{2}\right)y$, since

$$-\|u_s-z\|>-\|y-z\|,$$

we have for all $\alpha\geqq 1$,

$$\begin{aligned}\|u_s+\alpha(y-u_s)-z\| &= \|(1-\alpha)u_s+\alpha y-z\| \\ &= \|(1-\alpha)(u_s-z)+\alpha(y-z)\|\end{aligned}$$

$$\begin{aligned}&=\|(\alpha-1)(z-u_s)+\alpha(y-z)\|\\&\geqq\alpha\|y-z\|-(\alpha-1)\|z-u_s\|\\&\geqq\alpha\|y-z\|-(\alpha-1)\|y-z\|\\&=\|y-z\|.\end{aligned}$$

Hence, by Theorem 2.5 in [5], we have

$$\langle u_s+\alpha(y-u_s)-y, J(y-z)\rangle\geqq 0 \text{ for all } \alpha\geqq 1$$

and hence

$$\langle u_s-y, J(y-z)\rangle\leqq 0.$$

Therefore

$$\langle S(s+t_0)x-y, J(y-z)\rangle\leqq 0$$

for all $s\geqq 0$. Letting $t\geqq t_0$, then we have

$$\langle S(t)x-y, J(y-z)\rangle\leqq 0.$$

PROPOSITION 2.5. *Let X be a uniformly convex Banach space with a Fréchet differentiable norm and C a closed convex subset of X. Let $x\in C$ and $F(S)\neq\phi$. Then the set $W(x)\cap F(S)$ consists of at most one point.*

Proof. Let $y, z\in W(x)\cap F(S)$. Then, since $(y+z)/2$ in $F(S)$, it follows from Lemma 2.4 that there exists $t_0\geqq 0$ such that

$$\langle S(t)x-(y+z)/2, J((y+z)/2-z)\rangle\leqq 0$$

for all $t\geqq t_0$. Since $y\in\overline{\text{conv}}\{S(t)x : t\geqq t_0\}$, we have

$$\langle y-(y+z)/2, J((y+z)/2-z)\rangle\leqq 0$$

and hence

$$\langle (y-z)/2, J((y-z)/2)\rangle\leqq 0.$$

This implies that $y=z$.

Now, we prove lemmas which play a crucial role in the proof of our main theorem in the next section. The following Lemma 2.6 ([11]) is well known, and is an analogue of Lemma 4 in [10].

LEMMA 2.6. *Let C be a closed convex subset of a uniformly convex Banach space X with a Fréchet differentiable norm. Let $\{S(t) : t\geqq 0\}$ be* *a nonexpansive semigroup on C. If μ is a finite mean on* $[0,\infty)$, *then for all* $x\in C$,

$$\lim_{t\to\infty}\sup_{s\geqq 0}\|S(s)\mathcal{T}_\mu S(t)x-\mathcal{T}_\mu S(s+t)x\|=0.$$

LEMMA 2.7. *Let* X, C *and* $\{S(t): t\geqq 0\}$ *be as in Lemma* 2.6. *If* μ *is a compact mean on* $[0,\infty)$, *then for all* $x\in C$,

$$\lim_{t\to\infty}\sup_{s\geqq 0}\|S(s)\mathcal{T}_\mu S(t)x-\mathcal{T}_\mu S(s+t)x\|=0.$$

Proof. Since μ is a compact mean on $[0,\infty)$, there exists a finite mean λ on $[0,\infty)$ such that

$$\|\mathcal{T}_\mu S(t)x-\mathcal{T}_\lambda S(t)x\|<\frac{\varepsilon}{3}$$

for all $t\geqq 0$ ([11]). From Lemma 2.6, there exists $t_0\geqq 0$ such that for all $t\geqq t_0$,

$$\sup_{s\geqq 0}\|S(s)\mathcal{T}_\lambda S(t)x-\mathcal{T}_\lambda S(s+t)x\|<\frac{\varepsilon}{3}.$$

Therefore, we obtain for all $t\geqq t_0$,

$$\begin{aligned}
&\sup_{s\geqq 0}\|S(s)\mathcal{T}_\mu S(t)x-\mathcal{T}_\mu S(s+t)x\|\\
&\quad\leqq\sup_{s\geqq 0}\{\|S(s)\mathcal{T}_\mu S(t)x-S(s)\mathcal{T}_\lambda S(t)x\|\\
&\qquad+\|S(s)\mathcal{T}_\lambda S(t)x-\mathcal{T}_\lambda S(s+t)x\|\\
&\qquad+\|\mathcal{T}_\lambda S(s+t)x-\mathcal{T}_\mu S(s+t)x\|\}\\
&\quad\leqq\|\mathcal{T}_\mu S(t)x-\mathcal{T}_\lambda S(t)x\|\\
&\qquad+\sup_{s\geqq 0}\|S(s)\mathcal{T}_\lambda S(t)x-\mathcal{T}_\lambda S(s+t)x\|\\
&\qquad+\sup_{s\geqq 0}\|\mathcal{T}_\lambda S(s+t)x-\mathcal{T}_\mu S(s+t)x\|\\
&\quad<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.
\end{aligned}$$

Hence for all $x\in C$,

$$\lim_{t\to\infty}\sup_{s\geqq 0}\|S(s)\mathcal{T}_\mu S(t)x-\mathcal{T}_\mu S(s+t)x\|=0.$$

LEMMA 2.8. *Let* X, C *and* $\{S(t): t\geqq 0\}$ *be as in Lemma* 2.6 *and* $\{\mu_\alpha\}$

a net of compact means on $[0, \infty)$. *Suppose that* $\{\mathcal{T}_{\mu_\alpha} S(s_\alpha + h)x\}$ *converges weakly to* $p \in C$ *uniformly in* $h \geqq 0$ *for all* $x \in C$ *and* $s_\alpha \geqq 0$ *as* $\alpha \to \infty$. *Then for every c-invariant mean* η *on* $[0, \infty)$, $\mathcal{T}_\eta x = p$.

Proof. Since $\{\mu_\alpha\}$ is a net of compact means on $[0, \infty)$, for any $\varepsilon > 0$, there exists a finite mean λ_α for each μ_α such that

$$\|\mathcal{T}_{\lambda_\alpha} S(s)x - \mathcal{T}_{\mu_\alpha} S(s)x\| < \frac{\varepsilon}{2}$$

for all $s \geqq 0$ ([11]). For $x^* \in X^*$ with $\|x^*\| \leqq 1$, define $f : [0, \infty) \to R$ by $f(h) = \langle S(h)x, x^* \rangle$, which is continuous. From the hypothesis there exists α_0 such that

$$|\langle \mathcal{T}_{\mu_\alpha} S(s_\alpha + h)x, x^* \rangle - \langle p, x^* \rangle| < \frac{\varepsilon}{2}$$

for every $\alpha \geqq \alpha_0$ *and* $h \geqq 0$. Thus for all $h \geqq 0$,

$$\begin{aligned} &|\langle \mathcal{T}_{\lambda_\alpha} S(s_\alpha + h)x, x^* \rangle - \langle p, x^* \rangle| \\ &\quad \leqq |\langle \mathcal{T}_{\lambda_\alpha} S(s_\alpha + h)x - \mathcal{T}_{\mu_\alpha} S(s_\alpha + h)x, x^* \rangle| \\ &\qquad + |\langle \mathcal{T}_{\mu_\alpha} S(s_\alpha + h)x, x^* \rangle - \langle p, x^* \rangle| < \varepsilon. \end{aligned}$$

If $\lambda_\alpha = \sum_{i=1}^n a_i \delta_{s_i}$ ($s_i \geqq 0$, $a_i \geqq 0$, $i = 1, 2, 3 \cdots, n$, $\sum_{i=1}^n a_i = = 1$) for some $\alpha (\geqq \alpha_0)$, then

$$\begin{aligned} \langle \mathcal{T}_{\lambda_\alpha} S(s_\alpha + h)x, x^* \rangle &= \sum_{i=1}^n a_i \langle S(s_i + s_\alpha + h)x, x^* \rangle \\ &= \sum_{i=1}^n a_i f(s_i + s_\alpha + h) \\ &= (\sum_{i=1}^n a_i r_{s_i + s_\alpha} f)(h). \end{aligned}$$

Hence we have,

$$\sup_{h \geqq 0} |(\sum_{i=1}^n a_i r_{s_i + s_\alpha} f)(h) - \langle p, x^* \rangle| \leqq \varepsilon.$$

Therefore we have, for a c-invariant mean η,

$$\begin{aligned}|\eta(f)-\langle p, x^*\rangle| &= |\sum_{i=1}^{n} a_i\eta(r_{s_i+s_\alpha}f)-\langle p, x^*\rangle| \\ &= |\eta(\sum_{i=1}^{n} a_i r_{s_i+s_\alpha}f)-\langle p, x^*\rangle| \\ &= |\eta(\sum_{i=1}^{n} a_i r_{s_i+s_\alpha}f-\langle p, x^*\rangle\cdot 1)| \\ &\leqq \|\eta\|\sup_{h\geqq 0}|(\sum_{i=1}^{n} a_i r_{s_i+s_\alpha}f)(h)-\langle p, x^*\rangle| \\ &\leqq \varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\begin{aligned}\langle p, x^*\rangle=\eta(f) &= \int_0^\infty f(h)d\eta(h) \\ &= \int_0^\infty \langle S(h)x, x^*\rangle d\eta(h)=\langle \mathcal{T}_\eta x, x^*\rangle.\end{aligned}$$

Since $x^*(\|x^*\|\leqq 1)$ is arbitrary, we have $\mathcal{T}_\eta x=p$.

III. Main Result

Now, we can prove a nonlinear ergodic theorem for a nonexpansive semigroup in uniformly convex Banach spaces with a Fréchet differentiable norm.

In [2], R.E. Bruck proved the mean ergodic theorem for nonexpansive mappings.

THEOREM 3.1. *Let C be a bounded closed convex nonempty subset of a uniformly rotund Banach space X which has a Fréchet differentiable norm and $T: C\to C$ a nonexpansive mapping. Then the Cesàro mean of $\{T^n x\}$ converges weakly to a fixed point of T.*

In [10], N. Hirano also proved the following theorem.

THEOREM 3.2. *Let X be a uniformly convex Banach space which has a Fréchet differentiable norm. Let C be a closed convex subset of X and $T: C\to C$ a nonexpansive mapping. Then the following conditions are equivalent:*

(a) $F(T)\neq\phi$,

(b) $\{T^n x\}$ *is bounded for each x in C,*

(c) *For each* x *in* C, $S_nT^kx=\frac{1}{n}\sum_{i=0}^{n-1}T^{k+i}x$

converges weakly to y *in* $F(T)$ *uniformly in* $k=1, 2, \cdots$.

THEOREM 3. 3. *Let* C *be a closed convex subset of a uniformly convex Banach space* X *with a Fréchet differentiable norm and* $\{S(t) : t\geqq 0\}$ *a nonexpansive semigroup on* C. *If* $F(S)\neq\phi$, *then for all* $x\in C$,

$$A_tS(h)x=\frac{1}{t}\int_0^t S(s+h)xds$$

converges weakly to a point $p\in F(S)$ *uniformly in* $h\geqq 0$ *as* $t\to\infty$.

Proof. Let for all $x^*\in X^*$,

$$\langle A_tx, x^*\rangle=\frac{1}{t}\int_0^t\langle S(s)x, x^*\rangle ds$$

and let

$$\gamma_t=\frac{1}{t}\int_0^t\delta_s ds$$

for every $t>0$ where δ_s is a mean on $[0, \infty)$ defined by $\delta_s(f)=f(s)$ for all $f\in L^1{}_{\mathrm{loc}}([0, \infty))$. Then each γ_t is a continuous linear functional on $B([0, \infty))\cap L^1{}_{\mathrm{loc}}([0, \infty))$ and

$$\gamma_t(1)=1=\|\gamma_t\|.$$

Therefore, by the Hahn-Banach theorem, there exists a mean μ_t on $[0, \infty)$, which is an extension of γ_t and $\mathcal{T}_{\mu_t}x=A_tx$.

Let $K_t=[0, t]$ which is compact. Then we have

$$\mu_t(1_{K_t})=\gamma_t(1_{K_t})=1.$$

Hence μ_t is a compact mean on $[0, \infty)$. Therefore, from the Lemma 2. 7, there exists $s_t\geqq 0$ for each μ_t such that

$$\sup_{s\geqq 0}\|S(s)A_tS(v_t)x-A_tS(s+v_t)x\|<\frac{1}{t}$$

for all $v_t\geqq 0$ with $v_t\geqq s_t$. Furthermore, it is clear that

$$\|\mu_t-r_s{}^*\mu_t\|_c\to 0$$

as $t\to\infty$ for each $s\geqq 0$, where r_s^* is the conjugate operator of r_s which is a continuous linear operator in $B([0,\infty)$ for all $s\geq 0$ and $\|\cdot\|_c$ is the norm of $CB([0,\infty))$.

Let p be a limit point of $\{A_tS(s_t)x\}$ with respect to the weak topology. Next, for any $\varepsilon>0$ and $s\geqq 0$, taking t so large that $\frac{1}{t}<\frac{\varepsilon}{2}$ and

$$\|\mu_t-r_s^*\mu_t\|_c<\frac{\varepsilon}{2D},$$

where $D=\sup_{h\geqq 0}\|S(h)x\|$ (since $F(S)\neq\phi$), we have

$$\begin{aligned}
&\|A_tS(s_t)x-A_tS(s_t+s)x\| \\
&=\sup_{\|x^*\|\leqq 1}|\langle A_tS(s_t)x-A_tS(s_t+s)x, x^*\rangle| \\
&=\sup_{\|x^*\|\leqq 1}\left|\frac{1}{t}\int_0^t\langle S(s_t+h)x, x^*\rangle d\mu_t(h)\right. \\
&\quad\left.-\frac{1}{t}\int_0^t\langle S(s_t+h+s)x, x^*\rangle d\mu_t(h)\right| \\
&\leqq\sup_{\|x^*\|\leqq 1}(\|\mu_t-r_s^*\mu_t\|_c\cdot\sup_{h\geqq 0}|\langle S(s_t+h)x, x^*\rangle|) \\
&\leqq\|\mu_t-r_s^*\mu_t\|_c\cdot D \\
&<\frac{\varepsilon}{2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|A_tS(s_t)x-S(s)A_tS(s_t)x\| \\
&\quad\leqq\|A_tS(s_t)x-A_tS(s_t+s)x\| \\
&\qquad+A_tS(s_t+s)x-S(s)A_tS(s_t)x\| \\
&\quad\leqq\frac{\varepsilon}{2}+\frac{1}{t}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

This means that $\|A_tS(s_t)x-S(s)A_tS(s_t)x\|$ converges to 0 for all $s\geqq 0$. Since $I-S(s)$ is demiclosed, we have $S(s)p=p$ and $p\in F(S)$. It also follows from the assumption of μ_t that $p\in\bigcap_{h\geqq 0}\overline{\text{conv}}\ \{S(w)x : w\geqq h\}$. If not, then there exists $h_0\geqq 0$, $x^*\in X^*$ and $c>0$ such that

$$\langle p, x^*\rangle+c < \inf\{\langle z, x^*\rangle : z\in\overline{\text{conv}}\{S(w)x : w\geqq h_0\}\}.$$

Since p is a weak limit point of $\{A_tS(s_t)x\}$ and

$$\|\mu_t - r_{h_0}{}^*\mu_t\|_c \to 0,$$

there exists t_0 such that

$$|\langle p, x^*\rangle - \langle A_{t_0}S(s_{t_0})x, x^*\rangle| < \frac{c}{2}$$

and

$$\|\mu_{t_0} - r_{h_0}{}^*\mu_{t_0}\|_c < \frac{c}{2D\|x^*\|}.$$

So, we have

$$\begin{aligned}
\langle p, x^*\rangle + c &< \inf\{\langle z, x^*\rangle : z \in \overline{\text{conv}}\{S(w)x : w \geqq h_0\}\} \\
&\leqq \inf\{\langle S(w)x, x^*\rangle : w \geqq h_0\} \\
&\leqq \inf\{\langle S(w)x, x^*\rangle : w \geqq h_0 + s_{t_0}\} \\
&\leqq \frac{1}{t_0}\int_0^{t_0} \langle S(h_0 + s_{t_0} + w)x, x^*\rangle d\mu_{t_0}(w) \\
&= \langle A_{t_0}S(h_0 + s_{t_0})x, x^*\rangle \\
&= \langle A_{t_0}S(h_0 + s_{t_0})x - A_{t_0}S(s_{t_0})x, x^*\rangle \\
&\quad + \langle A_{t_0}S(s_{t_0})x, x^*\rangle \\
&\leqq \|x^*\|\|\mu_{t_0} - r_{h_0}{}^*\mu_{t_0}\|_c \cdot D + \langle p, x^*\rangle + \frac{c}{2} \\
&< \frac{c}{2} + \langle p, x^*\rangle + \frac{c}{2} = \langle p, x^*\rangle + c.
\end{aligned}$$

This is a contradiction. Hence,

$$p \in \bigcap_{h \geqq 0} \overline{\text{conv}}\{S(w)x : w \geqq h\} \cap F(S).$$

Since $\bigcap_{h \geqq 0} \overline{\text{conv}}\{S(w)x : w \geqq h\} \cap F(S)$ is a singleton set from Proposition 2.5, $\{A_tS(s_t)x\}$ converges weakly to $p \in F(S)$. Furthermore, by a quite similar argument, $\{A_tS(s_t + h)x\}$ converges weakly to $p \in F(S)$ uniformly in $h \geqq 0$ for all $x \in C$. And so, we obtain $\mathcal{T}_\eta x = p$ for all c-invariant mean η on $[0, \infty)$ from the Lemma 2.8.

Now, we shall show that $\{A_tS(h)x\}$ converges weakly to p uniformly in $h \geqq 0$. If we deny the assertion, then there exists $x^* \in X^*$, $\varepsilon > 0$, $t_\beta \geqq \beta$ and $h_\beta \geqq 0$ for all β such that

$$|\langle A_{t_\beta}S(h_\beta)x - p, x^*\rangle| \geqq \varepsilon.$$

We may assume, taking subnet, that $\eta_\beta = r_{h_\beta} * \mu_{t_\beta}$ converges to $\eta \in B([0,\infty))^*$ with respect to the weak star topology. Then η is a c-invariant mean on $[0,\infty)$. Hence we have

$$\begin{aligned}\langle A_{t_\beta} S(h_\beta)x, x^*\rangle &= \frac{1}{t_\beta}\int_0^{t_\beta} \langle S(h_\beta+h)x, x^*\rangle d\mu_{t_\beta}(h)\\ &= \frac{1}{t_\beta}\int_0^{t_\beta} \langle S(h)x, x^*\rangle dr_{h_\beta} * \mu_{t_\beta}(h)\\ &= \frac{1}{t_\beta}\int_0^{t_\beta} \langle S(h)x, x^*\rangle d\eta_\beta(h)\\ &\to \frac{1}{t_\beta}\int_0^{t_\beta} \langle S(h)x, x^*\rangle d\eta(h)\\ &= \langle \mathcal{T}_\eta x, x^*\rangle = \langle p, x^*\rangle.\end{aligned}$$

This is a contradiction. Hence

$$A_t S(h)x = \frac{1}{t}\int_0^t S(s+h)x\,ds$$

converges weakly to $p \in F(S)$ uniformly in $h \geqq 0$.

References

1. F.E. Browder, *Nonlinear operator and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math. 18, Amer. math. Soc. Providence, R.I. (1976).
2. R.E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. 32 (1979), 107–116.
3. J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.
4. M.M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
5. F.R. Deutsch and P.H. Maserick, *Applications of the Hahn-Banach theorem in approximation theory*, SIAM Rev. 9 (1967), 516–531.
6. J. Diestel, *Geometry of Banach spaces selected topic*, Lecture Notes in Math. Springer-Verlag. New York. 485 (1975).
7. T.M. Flett, *Differential analysis*, Cambridge Univ. Press (1980).
8. C.W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369–372.
9. V.I. Gurarii, *On the differential properties of the modulus of convexity in a Banach space (In Russian)*, Math. Issled, 2 (1967), 141–148.
10. N. Hirano, *A proof of the mean ergodic theorem for nonexpansive mappings in*

Banach spaces, Proc. Amer. Math. Soc. **78** (1980), 361-365.
11. N. Hirano, K. Kido and W. Takahashi, *Asymptotic behavior of commutative semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Anal. and Appl. **10** (1986), 229-249.
12. N. Hirano, and W. Takahashi, *Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space*, Pacific J. Math. **112** (1984), 333-346.
13. K. Kobayasi and I. Miyadera, *On the strong convergence of the Cesàro means of contractions in Banach spaces*, Proc. Japan Acad. **56** (1980), 245-249.
14. A.T. Lau, *Invariant means on almost periodic functions and fixed point properties*, Rocky Mountain, **3** (1973), 69-76.
15. A.T. Lau, *Invariant means on subsemigroup of locally compact groups*, Rocky Mountain, **3** (1973), 77-81.
16. A.T. Lau, and W. Takahashi, *Weak convergence and nonlinear ergodic theorems for reversible semigroup of nonexpansive mappings*, Pacific J. Math. (to appear).
17. T.C. Lim, *Asymptotic centers and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. **90** (1980), 135-143.
18. J.I. Milman, *Geometric theory of Banach space II, Geometry of the unit ball*, Uspehi Math. Nauk **26** (1971), 73-150.
19. A.T. Plant, *The differentiablity of nonlinear semigroup in uniformly convex spaces*, Israel J. Math. **38** (1981), 257-268.
20. S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274-276.
21. D. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl. **85** (1982), 172-178.
22. W. Takahashi, *Invariant functions for amenable semigroups of positive contractions on L^1*, Kódai Math. Sem. Rep. **23** (1971), 131-143.
23. T. Tigiel, *On the moduli of convexity and smoothness*, Studia Math. **56** (1976), 121-155.

Kyungnam University
Masan 610, Korea